96.48 A result on zigzag permutations: a combinatorial proof

Author(s): JONATHAN WEINSTEIN

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COLIN FOSTER

King Henry VIII School, Warwick Road, Coventry CV3 6AQ

e-mail: *c@foster77.co.uk*

96.48 A result on zigzag permutations: a combinatorial proof

Generating functions are an extremely powerful analytic tool for proving combinatorial identities, but combinatorial proofs typically provide a greater sense of insight and satisfaction. Finding a bijection, a direct correspondence between two sets, shows that they have the same size through something more than cosmic coincidence. Accordingly, Lewis [1], in this *Gazette*, after discovering a striking identity by matching generating functions, asked whether the identity might have a direct combinatorial proof. I shall provide one here.

Fix an integer n and let S_n be the set of permutations of $\{1, 2, \ldots, n\}$. It will be convenient throughout to write $\sigma(0) = \sigma(n + 1) = n + 1$ for all $\sigma \in S_n$. With this convention, the *zigzag permutations*, those which alternate upward and downward steps (beginning upward), can be defined by

$$Z_n = \left\{ \sigma \in S_n : \sigma(k) < \min\big(\sigma(k - 1), \sigma(k + 1)\big), k = 1, 3, 5, \ldots \right\}. \quad (1)$$

That is, all of the odd-indexed values should be 'valleys' in a graph of the permutation. I frame the definition this way because violations of this 'odd-valley rule' play a key role in the later proof. We let $Y_n = S_n - Z_n$ be the non-zigzag permutations. Also, let O_n be the permutations with an odd number of upward steps, i.e.

$$O_n = \left\{ \sigma \in S_n : \big|\{i \in \{1, \ldots, n - 1\} : \sigma(i) < \sigma(i + 1)\}\big| \text{ is odd} \right\}$$

and let $E_n = S_n - O_n$. (These sets bear no relation to the usual odd and even permutations.) Lewis found this surprising relationship:

Theorem 1: For odd n, $|Z_n| = \big||O_n| - |E_n|\big|$.

In attacking this result, I first noticed an easy proof that for *even* n, $|O_n| = |E_n|$. The operation θ of reversing a permutation (i.e. $\theta(\sigma) = \sigma \circ \tau$ where $\tau(i) = n + 1 - i$) changes the direction of each of the $n - 1$ steps and is self-inverse, so it forms a bijection between O_n and E_n. This led me to hope that for odd n I could find a map ϕ which reverses an even-length subsequence of each $\sigma \in Y_n$. This would, ideally, pair off the non-zigzag permutations into odd-even pairs, showing that $|Y_n \cap O_n| = |Y_n \cap E_n|$. The zigzag permutations themselves all have the same sign, so this would complete the proof. The challenge was to choose the subsequence so that the directions of the steps at its boundary are unaffected by the reversal, and so

that the same subsequence will be selected from $\phi(\sigma)$ as from σ, making ϕ self-inverse. That is, whichever landmarks are used to select the subsequence must be unaffected by the reversal. I now show that this can be accomplished, by using 'high-elevation' landmarks whose prominence is unaffected by reversing lower-lying elements.

Proof of Theorem 1: Given $\sigma \in Y_n$, define $\phi(\sigma)$ by the following algorithm:

1. Let k be the odd integer violating (1) for which $\sigma(k)$ is largest.
2. We subdivide into two cases, according to whether the violation occurs between k and $k + 1$ or between k and $k - 1$. (If both, we arbitrarily give precedence to the former.)
    (a) If $\sigma(k + 1) < \sigma(k)$, let l be the smallest index among $k + 2, \ldots, n + 1$ for which $\sigma(l) > \sigma(k)$.
    (b) Otherwise (so that $\sigma(k - 1) < \sigma(k)$), let l be the largest index among $0, \ldots, k - 2$ for which $\sigma(l) > \sigma(k)$.
3. Form $\phi(\sigma)$ from σ by reversing the subsequence strictly between indices k and l.

Now we make the following observations:

*   If l were odd, it would violate (1), which would contradict the definition of k because $\sigma(l) > \sigma(k)$. Hence l is even[*], and $k - l$ is odd. (This includes the cases $l = 0$ and $n + 1$; here is where we use the fact that n is odd.) Also $|k - l| > 1$.
*   Reversing the indicated subsequence switches the direction of each of the $|k - l| - 2$ internal steps, without affecting those at the boundary (since $\sigma(k)$ and $\sigma(l)$ are greater than all elements between them). Therefore, ϕ maps elements of E_n to O_n and vice versa.
*   The reversal does not affect the defining properties of k or l (since the elements being reversed are smaller than $\sigma(k)$ or $\sigma(l)$). This ensures that $\phi(\sigma) \in Y_n$, and that $\phi(\phi(\sigma)) = \sigma$.
*   We conclude that ϕ gives a bijection between $Y_n \cap O_n$ and $Y_n \cap E_n$, completing the proof via the simple calculation

$$\left|\,|O_n| - |E_n|\,\right| = \left|\,|Y_n \cap O_n| + |Z_n \cap O_n| - |Y_n \cap E_n| - |Z_n \cap E_n|\,\right|$$

$$= \left|\,|Z_n \cap O_n| - |Z_n \cap E_n|\,\right|$$

$$= |Z_n|$$

    where the last step holds because (for any fixed n) the zigzag permutations all have the same number of upward steps, hence are either all odd or all even.

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[*]   This includes the cases $l = 0$ or $n + 1$; here is where we use the fact that n is odd.

Example 1: If $\sigma^{-1}(n)$ is odd, we always have $k = \sigma^{-1}(n)$, $l = n + 1$. In this case ϕ simply reverses the sequence after the appearance of n. So for $n = 5$, $\phi(5,2,3,4,1) = (5,1,4,3,2)$ (and vice versa), where $k = 1$, $l = 6$.

Example 2: For $\sigma = (3,2,4,5,1)$, we enter case (2b) and have $k = 3$, $l = 0$, giving $\phi(\sigma) = (2,3,4,5,1)$.

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JONATHAN WEINSTEIN
Kellogg School of Management, Evanston, IL, USA 60208-2001

96.49 On the diagonals of a Stirling number triangle

Stirling numbers come in two varieties, namely those of the first and second kind. In this note we are interested only in the latter. The *Stirling numbers of the second kind* $S(n, k)$ enumerate the partitions of a set of n labelled objects, $\{1, 2, \dots, n\}$ say, into exactly k non-empty disjoint parts. For example, the partitions of $\{1, 2, 3, 4\}$ into exactly 3 non-empty parts are given by

$$\{\{1\}, \{2\}, \{3, 4\}\}, \quad \{\{1\}, \{3\}, \{2, 4\}\}, \quad \{\{1\}, \{4\}, \{2, 3\}\},$$
$$\{\{2\}, \{3\}, \{1, 4\}\}, \quad \{\{2\}, \{4\}, \{1, 3\}\}, \quad \{\{3\}, \{4\}, \{1, 2\}\},$$

from which we see that $S(4, 3) = 6$. Table 1 gives the Stirling numbers of second kind up to $n = 7$. Note that, since $S(n, k) = 0$ when $k > n$, we refer to this as a 'number triangle'.

n	$S(n, 1)$	$S(n, 2)$	$S(n, 3)$	$S(n, 4)$	$S(n, 5)$	$S(n, 6)$	$S(n, 7)$
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1

TABLE 1: Stirling numbers of the second kind $S(n, k)$

It is well known [1] that the exponential generating function of the sequence in the kth column of Table 1 is given by

$$F_k(x) = \frac{1}{k!}(e^x - 1)^k.$$

In other words, the nth term of the kth column is the coefficient of $x^n / n!$ in $F_k(x)$, which is

$$\frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$