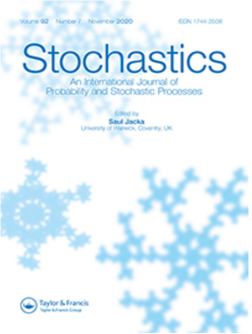

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On solutions of equations with measurable coefficients driven by α- stable processes

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ABSTRACT

We prove the existence of solutions for the stochastic differential equation $\mathrm{d}X_t = b(t, X_{t-})\,\mathrm{d}Z_t + a(t, X_t)\,\mathrm{d}t, X_0 \in \mathbb{R}, t \geq 0$, with the measurable coefficients a and b satisfying the condition $0 < \mu \leq |b(t,x)| \leq \nu$ and $|a(t,x)| \leq K$ for all $t \geq 0, x \in \mathbb{R}$, where μ, ν, and K are constants. The driving process Z is a symmetric stable process of index $1 < \alpha < 2$. This generalizes the result of Krylov [*Controlled Diffusion Processes*, Springer, New York, 1980] for the case of $\alpha = 2$, that is, when Z is a Brownian motion. The proof is based on integral estimates of the Krylov type for the given equation, which are also derived in this note and are of independent interest. Moreover, unlike in Krylov [*Controlled Diffusion Processes*, Springer, New York, 1980], we use a different approach to derive the corresponding integral estimates.

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1. Introduction

We consider here a stochastic differential equation of the form

$$\mathrm{d}X_t = b(t, X_{t-})\,\mathrm{d}Z_t + a(t, X_t)\,\mathrm{d}t, \quad X_0 = x_0 \in \mathbb{R},\ t \geq 0. \tag{1}$$

For the case when the driving process Z is a Brownian motion, the existence of solutions for Equation (1) with *measurable coefficients* a and b was first established by Krylov in [7]. His proof was based on corresponding integral estimates for solutions X of (1), which he was also the first to derive. Those integral estimates turned out later to be very useful in various areas of stochastic processes, including the optimal control of processes described by Equation (1). Estimates of this kind are often referred to as *Krylov type estimates*.

In order to prove the corresponding integral estimates, Krylov used the Bellman principle of optimality, known in the control theory of stochastic processes. Given a smooth function $f(t,x), (t,x) \in \mathbb{R}^2$, he considered the value function

$$v(t,x) := \sup_{\beta \in \mathcal{B}} \mathbf{E} \int_0^\infty e^{-\phi_s^\beta}\, \psi_s^\beta f(t + r_s^\beta, x + X_s^\beta)\,\mathrm{d}s, \tag{2}$$

where (ϕ^β, ψ^β) and (r^β, X^β) are appropriately chosen stochastic processes and $\mathcal{B}$ is a suitably chosen set of control parameters.

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Krylov derived the corresponding Bellman equation for the function $v(t,x)$, and upon integrating it he received estimates of the form

$$\sup_{(t,x)\in\mathbb{R}^2} |v(t,x)| \le M\|f\|_{L_p}, \tag{3}$$

where $\|f\|_{L_p} := (\int_{\mathbb{R}^2} |f(t,x)|^p \, \mathrm{d}t\, \mathrm{d}x)^{1/p}, p \in [1,\infty)$, is the L_p-norm of the function f. Finally, using Itó's formula and the estimates (3), he obtained integral estimates of the form

$$\mathbf{E}\int_0^\infty f(s,X_s)\,\mathrm{d}s \le M\|f\|_{L_p}, \tag{4}$$

known now as Krylov type estimates.

As an application of (4), Krylov proved the existence of solutions of Equation (1) in the case when Z is a Brownian motion and the measurable coefficients a and b are such that, for all $(t,x) \in [0,\infty)\times\mathbb{R}$, it holds that

$$0 < \mu \le |b(t,x)| \le \nu, \quad |a(t,x)| \le K \tag{5}$$

for some constants μ, ν, and K.

In this note we consider Equation (1) when the driving process Z is a symmetric stable process of index $1 < \alpha \le 2$. For $\alpha = 2$, Z is a Brownian motion process.

One of the main results here is the proof of the existence of solutions of Equation (1) when the coefficients a and b are measurable and satisfy the condition (5). This extends the result of Krylov for the Brownion motion case to the case of a symmetric stable process with index $1 < \alpha \le 2$.

The coefficients $a(t,x)$ and $b(t,x)$ of stochastic Equation (1) are defined only on $[0,\infty)\times\mathbb{R}$. However, it will be convenient for us later to work on the space $\mathbb{R}^2$ instead of its subset $[0,\infty)\times\mathbb{R}$. For that reason, we do extend a and b to functions $\bar{a}$ and $\bar{b}$, respectively, in the following way:

$$\bar{a}(t,x) := \begin{cases} a(-t,x), & (t,x) \in (-\infty,0)\times\mathbb{R}, \\ a(t,x), & (t,x) \in [0,\infty)\times\mathbb{R}, \end{cases}$$

and

$$\bar{b}(t,x) := \begin{cases} b(-t,x), & (t,x) \in (-\infty,0)\times\mathbb{R}, \\ b(t,x), & (t,x) \in [0,\infty)\times\mathbb{R}. \end{cases}$$

It is clear that functions $\bar{a}$ and $\bar{b}$ satisfy the condition (5) if and only if the functions a and b satisfy that condition.

Let (X,Z) be a solution of Equation (1) on a probability space $(\Omega,\mathcal{F},\mathbf{P})$. Since, for any $t \ge 0$,

$$\int_0^t \bar{b}(s,X_{s-})\,\mathrm{d}Z_s = \int_0^t b(s,X_{s-})\,\mathrm{d}Z_s$$

and

$$\int_0^t \bar{a}(s,X_s)\,\mathrm{d}s = \int_0^t a(s,X_s)\,\mathrm{d}s$$

P-a.s., it follows that the pair (X, Z) solves the equation

$$\mathrm{d}X_t = \bar{b}(t, X_{t-})\,\mathrm{d}Z_t + \bar{a}(t, X_t)\,\mathrm{d}t, \quad X_0 = x_0 \in \mathbb{R},\ t \geq 0, \tag{6}$$

on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ as well. The converse is obviously also true.

To prove the existence of solutions of Equation (1), we will first derive the corresponding Krylov type estimates for processes X satisfying Equation (6). However, unlike in [7], we do not use any facts from the optimal control theory for stochastic processes but instead consider a parabolic integro-differential equation of the form

$$u_t + |\bar{b}|^\alpha \mathcal{L}u + \bar{a}u_x - \lambda(1 + |\bar{b}|^\alpha)u + f = 0 \text{ a.e. in } \mathbb{R}^2, \tag{7}$$

where $\mathcal{L}$ is the generator of the process Z (see definitions below), λ is a fixed positive constant, and u_t, u_x are partial derivatives of u in t and x, respectively.

To be more precise, we will consider Equation (7) for such values of $\lambda > 0$, so that

$$\mu^\alpha\left(\lambda + \frac{1}{2}|x|^\alpha\right)^2 \geq \frac{4K^2}{\mu^\alpha}x^2 \tag{8}$$

for all $x \in \mathbb{R}$.

Since $\alpha \in (1, 2)$, it is clear that there exists $\lambda_0 > 0$ such that (8) is satisfied for all $\lambda \in [\lambda_0, \infty)$. We also note that any value of λ satisfying (8) depends on μ, K, and α only.

Assuming that the functions a and b satisfy the condition (5), we will prove some important *a priori estimates* for Equation (7), of the form

$$\|u\|_{L_2} + \|u_t\|_{L_2} + \|\mathcal{L}u\|_{L_2} \leq M\|f\|_{L_2}, \tag{9}$$

which, in turn, will imply the estimates

$$\sup_{(t,x)\in\mathbb{R}^2} |u(t,x)| \leq M\|f\|_{L_2}.$$

Moreover, a priori estimates (9) are then also used to prove the existence of a solution u of Equation (7) given a fixed function $f \in C_c^\infty(\mathbb{R}^2)$[1] and a fixed value of λ satisfying condition (8). The latter fact is important to derive the corresponding integral estimates.

Finally, we give a brief overview of existence results for Equation (1) with measurable coefficients a and b and $1 < \alpha < 2$ known for some particular cases.

Zanzotto [14] studied Equation (1) without drift (that is, when $a = 0$) and with time-independent coefficient b. The approach in [14] was a systematic use of the method of random time change.

The time-dependent Equation (1) without drift was studied by Pragarauskas and Zanzotto [12]. To prove the existence of solutions, they used the method of integral estimates similar to [7]. The corresponding integral estimates were proven by Pragarauskas in [11]. Engelbert and Kurenok [5] studied the time-dependent Equation (1) without drift and with $0 < \alpha < 2$, and they found different sufficient existence conditions for solutions. Their approach relied on time change techniques.

In [9], the author proved the existence of solutions for Equation (1) with time-independent and measurable coefficients a and b satisfying the condition

$$0 < \mu \leq |b(x)| \leq \nu, \quad |a(x)| \leq K,$$

for all $x \in \mathbb{R}$ and the constants μ, ν, and K.

2. Some preliminary facts

As usual, by $\mathbf{D}_{[0,\infty)}(\mathbb{R})$ we denote the Skorokhod space, i.e. the set of all real-valued functions $z : [0, \infty) \to \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *càdlàg* functions). For simplicity, we shall write $\mathbf{D}$ instead of $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbf{D}$ with the σ-algebra $\mathcal{D}$ of Borel subsets of $\mathbf{D}$ in the Skorokhod topology. By $\mathbf{D}^n$ we denote the n-dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \cdots \times \mathbf{D}$, with the corresponding σ-algebra $\mathcal{D}^n$ being the direct product of n one-dimensional σ-algebras $\mathcal{D}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)$. We use the notation $(Z, \mathbb{F})$ to indicate that a process Z is adapted to $\mathbb{F}$. A process $(Z, \mathbb{F})$ is called a symmetric stable process of index $\alpha \in (0, 2]$ if the trajectories of Z are càdlàg functions and $\mathbf{E}(\exp(i\xi(Z_t - Z_s))|\mathcal{F}_s) = \exp(-(t-s)c|\xi|^\alpha)$ for all $0 \le s < t$ and $\xi \in \mathbb{R}$, where $c > 0$ is a constant. The function $\psi(\xi) = c|\xi|^\alpha$ is called the characteristic exponent of the process Z.

The process Z is a process with independent increments, and thus is a Markov process. For any bounded measurable function $u : \mathbb{R} \to \mathbb{R}$ and $t \ge 0$, the operator

$$(T_t u)(x) := \mathbf{E}\Big(u(x + Z_t)\Big), \quad x \in \mathbb{R},$$

is the semigroup of convolution operators associated with Z. We can introduce the so-called infinitesimal generator $\mathcal{L}$ of the process Z as

$$(\mathcal{L}u)(x) = \lim_{t \downarrow 0} \frac{(T_t u)(x) - u(x)}{t}, \quad u \in D(\mathcal{L}), \tag{10}$$

where the domain $D(\mathcal{L})$ of $\mathcal{L}$ consists of all bounded measurable real functions u such that the limit in (10) exists uniformly.

It is known (see, e.g. [13], section 4.1) that for $\alpha < 2$

$$(\mathcal{L}u)(x) = \int_{\mathbb{R}\setminus\{0\}} [u(x+z) - u(x) - \mathbf{1}_{\{|z|<1\}} u'(x) z] \frac{c_1}{|z|^{1+\alpha}} \, \mathrm{d}z \tag{11}$$

for any $u \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ is the set of all bounded and twice continuously differentiable functions $u : \mathbb{R} \to \mathbb{R}$ whose derivatives are also bounded. We shall assume from now on the constant c_1 is chosen so that $\psi(\xi) = 1/2|\xi|^\alpha$. In the case of $\alpha = 2$, the infinitesimal generator of Z is the second derivative operator, that is, $\mathcal{L}u(x) = \frac{1}{2}u''(x)$.

On the other hand, in the case of $\alpha \in (0, 2)$, the process Z as a purely discontinuous Markov process can be described by its Poisson jump measure (the jump measure of Z on interval $[0, t]$) defined as

$$N(U \times [0, t]) = \sum_{s \le t} 1_U(Z_s - Z_{s-}).$$

The above equation describes the number of times before the time t that Z has jumps whose size lies in the set $U \in \mathbb{R} \setminus 0$. The corresponding Lévy measure of N is given by

$$\hat{N}(U) = \mathbf{E}N(U \times [0, 1]) = \int_U \frac{c_1}{|z|^{1+\alpha}} \, \mathrm{d}z, \quad U \in \mathbb{R} \setminus 0.$$

We recall that, for any $u \in L_1(\mathbb{R}^2)$, there exists its Fourier transform Fu defined as

$$[Fu](\tau, w) = \int_{\mathbb{R}^2} e^{is\tau} e^{ixw} u(s,x)\,\mathrm{d}s\,\mathrm{d}x, \quad (\tau, w) \in \mathbb{R}^2.$$

Moreover, if $Fu \in L_1(\mathbb{R}^2)$, then also the inverse Fourier transform F^{-1} of the function Fu exists, and

$$u(s,x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} [Fu](\tau, w) e^{-is\tau} e^{-ixw}\,\mathrm{d}\tau\,\mathrm{d}w, \quad (s,x) \in \mathbb{R}^2. \tag{12}$$

Clearly, calculating the Fourier transform of a function of two variables can be performed by first calculating the single Fourier transform in one variable and then in the other, in any order.

Next, we extend the operator $\mathcal{L}$, acting on suitable functions $u(t,x)$, $(t,x) \in \mathbb{R}^2$, in the following way. For any *fixed* t, we define

$$(\mathcal{L}u(t,\cdot))(x) = \int_{\mathbb{R}\setminus\{0\}} [u(t,x+z) - u(t,x) - \mathbf{1}_{\{|z|<1\}} u_x(t,x) z] \frac{c_1}{|z|^{1+\alpha}}\,\mathrm{d}z. \tag{13}$$

The following statement will be used frequently later.

Proposition 2.1: *Let* $0 < \alpha \le 2$ *and* $u \in S(\mathbb{R}^2)$, *where* $S(\mathbb{R}^2)$ *is the Schwarz space of rapidly decreasing functions. Then, it holds that*

(a) $F[\mathcal{L}u](\tau, w) = -\frac{1}{2}|w|^\alpha F[u](\tau, w)$;
(b) $F[u_t](\tau, w) = -i\tau F[u](\tau, w))$;
(c) $F[u_x](\tau, w) = -iwF[u](\tau, w))$.

Proof: We calculate

$$F[\mathcal{L}u](\tau, w) = \int_{\mathbb{R}} e^{i\tau t} \Big(\int_{\mathbb{R}} e^{ixw} (\mathcal{L}u(t,\cdot)(x)\,\mathrm{d}x \Big)\,\mathrm{d}t. \tag{14}$$

The inner integral in (14) is the Fourier transform of the function $\mathcal{L}(u(t,\cdot))$ in the variable x where t is fixed. For any fixed t, the function $u(t,\cdot)$ belongs to the space $S(\mathbb{R})$ so that the inner integral is equal to

$$-\frac{1}{2}|w|^\alpha F_x[u](t,w),$$

where $F_x[u]$ is the Fourier transform of $u(t,x)$ in variable x (cf. Applebaum [2], Theorem 3.3.3). This proves statement (a).

The relations (b) and (c) follow easily by using partial integration. ■

Finally, let us introduce the following space of functions associated with the infinitisimal operator $\mathcal{L}$ of a symmetric stable process of index α. For any $u \in C_c^\infty(\mathbb{R}^2)$, define the norm

$$\|u\|_H := \|u\|_{L_2} + \|u_t\|_{L_2} + \|\mathcal{L}u\|_{L_2}, \tag{15}$$

where the right-hand side in (15) is finite. The finiteness of norms $\|u\|_{L_2}$ and $\|u_t\|_{L_2}$ is obvious. Moreover, by Proposition 2.1 and Plancherel's identity, $\|\mathcal{L}u\|_{L_2} = \|F(\mathcal{L}u)\|_{L_2} =$

$\||w|^\alpha F(u)\|_{L_2}$. Since $F(u) \in S(\mathbb{R}^2)$, it follows that $|w|^\alpha F(u) \in S(\mathbb{R}^2)$, and since $S(\mathbb{R}^2)$ is a subspace of $L_2(\mathbb{R}^2)$, it implies $\mathcal{L}u \in L_2(\mathbb{R}^2)$.

We say that a function $u \in L_2(\mathbb{R}^2)$ belongs to the space $H(\mathbb{R}^2)$ if there is a sequence of functions $u^n \in C_c^\infty(\mathbb{R}^2)$ such that

$$\|u^n\|_H < \infty \quad \text{for all } n = 1, 2, \dots,$$
$$\|u^n - u\|_{L_2} \to 0 \text{ as } n \to \infty,$$

and

$$\|u_t^n - u_t^m\|_{L_2} \to 0, \quad \|\mathcal{L}u^n - \mathcal{L}u^m\|_{L_2} \to 0 \text{ as } n, m \to \infty.$$

Any such sequence of functions u^n is called a defining sequence for u. The space H is then called a *Sobolev space*. The functions u_t and u_x in Equation (7) are understood as *generalized derivatives* of u in the variables t and x, correspondingly.

3. Analytic a priori estimates

In this section we consider the integro-differential equation of parabolic type (7) in the Sobolev space H with the norm $\|\cdot\|_H$ defined in (15). We assume that $\alpha \in (1, 2)$, that the coefficients a and b satisfy the condition (5), and that a fixed value of λ exists such that the inequality (8) holds.

We are interested in deriving some *a priori estimates* for a solution u of Equation (7) in terms of the L_2-norm of the function f. Since the existence of a solution is not known yet, such estimates are called a priori estimates. These estimates will be crucial in Section 4 for deriving integral estimates of the Krylov type for processes X satisfying stochastic Equation (6).

Moreover, the a priori estimates obtained here can be used to prove the existence of a solution $u \in H(\mathbb{R}^2)$ of Equation (7) for any $f \in C_c^\infty(\mathbb{R}^2)$ and any λ satisfying the condition (8). The corresponding proof is based on the *method of continuity* and the *method of a priori estimates* known in the theory of classical elliptic and parabolic equations; that is, $\mathcal{L}$ is the second derivative operator. The proof of the existence of a solution of Equation (7) is provided in the Appendix.

Lemma 3.1: *Let $u \in C_c^\infty(\mathbb{R}^2)$ be a solution of Equation (7) with $f \in L_2(\mathbb{R}^2)$. Then there are constants M_1 and M_2 such that*

$$\|u\|_H \le M_1 \|f\|_{L_2} \tag{16}$$

and

$$\sup_{(t,x)\in\mathbb{R}^2} |u(t,x)| \le M_2 \|f\|_{L_2}, \tag{17}$$

where the values of M_1 and M_2 depend on ν, μ, K, and α only.

Proof: It follows from (7) that

$$[(u_t - \lambda u) + |\bar{b}|^\alpha(\mathcal{L}u - \lambda u)]^2 = (\bar{a}u_x + f)^2 \le 2\bar{a}^2 u_x^2 + 2f^2$$

and

$$\frac{1}{|\bar{b}|^\alpha}(u_t - \lambda u)^2 + 2(u_t - \lambda u)(\mathcal{L}u - \lambda u) + |\bar{b}|^\alpha(\mathcal{L}u - \lambda u)^2 \le \frac{2}{|\bar{b}|^\alpha}(K^2 u_x^2 + f^2).$$

The condition (5) implies that

$$\frac{1}{\nu^\alpha}(u_t - \lambda u)^2 + 2(u_t - \lambda u)(\mathcal{L}u - \lambda u) + \mu^\alpha(\mathcal{L}u - \lambda u)^2 \le \frac{2}{\mu^\alpha}(K^2 u_x^2 + f^2). \tag{18}$$

We note further that $u \in S(\mathbb{R}^2)$, since $C_c^\infty(\mathbb{R}^2)$ is a subspace of $S(\mathbb{R}^2)$. Using Plancherel's identity and Proposition 2.1, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\Big(u_t(s,y) - \lambda u(s,y)\Big)^2 \mathrm{d}s\,\mathrm{d}y &= \int_{\mathbb{R}^2}\Big|F[u_t - \lambda u](\tau,w)\Big|^2 \mathrm{d}\tau\,\mathrm{d}w \\ &= \int_{\mathbb{R}^2}|F[u](\tau,w)|^2(\lambda^2 + \tau^2)\,\mathrm{d}\tau\,\mathrm{d}w, \quad (19)\\ \int_{\mathbb{R}^2}\Big(\mathcal{L}u(s,y) - \lambda u(s,y)\Big)^2 \mathrm{d}s\,\mathrm{d}y &= \int_{\mathbb{R}^2}|F[\mathcal{L}u - \lambda u](\tau,w)|^2\,\mathrm{d}\tau\,\mathrm{d}w \\ &= \int_{\mathbb{R}^2}|F[u](\tau,w)|^2\left(\lambda + \frac{1}{2}|w|^\alpha\right)^2 \mathrm{d}\tau\,\mathrm{d}w, \quad (20)\end{aligned}$$

and

$$\int_{\mathbb{R}^2} u_x^2(s,y)\,\mathrm{d}s\,\mathrm{d}y = \int_{\mathbb{R}^2}|F[u_x](\tau,w)|^2\,\mathrm{d}\tau\,\mathrm{d}w = \int_{\mathbb{R}^2}|w|^2|F[u](\tau,w)|^2\,\mathrm{d}\tau\,\mathrm{d}w. \tag{21}$$

Now, we integrate inequality (18) over $\mathbb{R}^2$ and use identities (19)–(21) and (8) to obtain

$$\begin{aligned}&\frac{1}{\nu^\alpha}\int_{\mathbb{R}^2}|F[u](\tau,w)|^2(\lambda^2 + \tau^2)\,\mathrm{d}\tau\,\mathrm{d}w + 2\int_{\mathbb{R}^2}\Big((u_t - \lambda u)(s,y)\Big)\Big((\mathcal{L}u - \lambda u)(s,y)\Big)\,\mathrm{d}s\,\mathrm{d}y \\ &+ \frac{\mu^\alpha}{2}\int_{\mathbb{R}^2}\left(\lambda + \frac{1}{2}|w|^\alpha\right)^2|F[u](\tau,w)|^2\,\mathrm{d}\tau\,\mathrm{d}w \le \frac{2}{\mu^\alpha}\int_{\mathbb{R}^2} f^2(s,y)\,\mathrm{d}s\,\mathrm{d}y. \quad (22)\end{aligned}$$

The last inequality implies

$$\begin{aligned}&\frac{\lambda^2}{\nu^\alpha}\int_{\mathbb{R}^2}|F[u](\tau,w)|^2\,\mathrm{d}\tau\,\mathrm{d}w + 2\int_{\mathbb{R}^2}\Big((u_t - \lambda u)(s,y)\Big)\Big((\mathcal{L}u - \lambda u)(s,y)\Big)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + \frac{\mu^\alpha\lambda^2}{2}\int_{\mathbb{R}^2}|F[u](\tau,w)|d\tau\,\mathrm{d}w^2 \le \frac{2}{\mu^\alpha}\int_{\mathbb{R}^2} f^2(s,y)\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

or

$$\begin{aligned}&\left(\frac{\mu^\alpha\lambda^2}{2} + \frac{\lambda^2}{\nu^\alpha}\right)\|u\|_{L_2}^2 + 2\int_{\mathbb{R}^2}\Big((u_t - \lambda u)(s,y)\Big)\Big((\mathcal{L}u - \lambda u)(s,y)\Big)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad \le \frac{2}{\mu^\alpha}\int_{\mathbb{R}^2} f^2(s,y)\,\mathrm{d}s\,\mathrm{d}y. \quad (23)\end{aligned}$$

To estimate the second term on the left-hand side of inequality (23), we first notice that its value is a real number. Using again then Plancherel's identity yields

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\Big((u_t-\lambda u)(s,y)\Big)\Big((\mathcal{L}u-\lambda u)(s,y)\Big)\,\mathrm{d}s\,\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^2}\overline{F[u_t-\lambda u](\tau,w)}\times F[\mathcal{L}u-\lambda u](\tau,w)\,\mathrm{d}\tau\,\mathrm{d}w\\
&\quad=Re\Big[\int_{\mathbb{R}^2}(\lambda-i\tau)\left(\lambda+\frac{1}{2}|w|^{\alpha}\right)|F[u]|^2(\tau,w)\,\mathrm{d}\tau\,\mathrm{d}w\Big]\\
&\quad=\int_{\mathbb{R}^2}\lambda\left(\lambda+\frac{1}{2}|w|^{\alpha}\right)|F[u](\tau,w))|^2\,\mathrm{d}\tau\,\mathrm{d}w\\
&\quad\geq\int_{\mathbb{R}^2}\lambda^2|F[u](\tau,w))|^2\,\mathrm{d}\tau\,\mathrm{d}w=\lambda^2\|u\|_{L_2}^2\geq 0.
\end{aligned}
$$

We have shown that

$$
\left(\frac{\mu^{\alpha}\lambda^2}{2}+\frac{\lambda^2}{\nu^{\alpha}}+\lambda^2\right)\|u\|_{L_2}^2\leq\frac{2}{\mu^{\alpha}}\|f\|_{L_2}^2,
$$

or

$$
\|u\|_{L_2}\leq M\|f\|_{L_2}, \tag{24}
$$

where the constant M only depends on μ,ν,K, and α.

Obviously,

$$
\|\mathcal{L}u\|_{L_2}\leq\|\mathcal{L}u-\lambda u\|_{L_2}+\lambda\|u\|_{L_2},
$$

and

$$
\|u_t\|_{L_2}\leq\|u_t-\lambda u\|_{L_2}+\lambda\|u\|_{L_2},
$$

so that estimate (16) follows then from (24), the inequality (22), and the established fact that the second term on the left-hand side of (18) is non-negative.

To prove estimate (17), we first notice that $F[u]\in L_1(\mathbb{R}^2)$, since $u\in S(\mathbb{R}^2)$. Using the Fourier inversion formula and the Cauchy-Schwarz inequality, we estimate

$$
\begin{aligned}
&|u(t,x)|^2\\
&\leq\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|F[u]|(\tau,w)\,\mathrm{d}\tau\,\mathrm{d}w\Big)^2\\
&=\frac{1}{16\pi^4}\Big(\int_{\mathbb{R}^2}|F[u](\tau,w)|\Big(\Big|-2\lambda-i\tau-\frac{1}{2}\Big|w|^{\alpha}|\Big)\Big(\Big|-2\lambda-i\tau-\frac{1}{2}\Big|w|^{\alpha}|\Big)^{-1}\mathrm{d}\tau\,\mathrm{d}w\Big)^2\\
&\leq\frac{1}{16\pi^4}I_1I_2,
\end{aligned}
$$

where

$$
I_1=\int_{\mathbb{R}^2}|F[u]|^2(\tau,w)\left|-2\lambda-i\tau-\frac{1}{2}\right|w|^{\alpha}|^2\,\mathrm{d}\tau\,\mathrm{d}\omega
$$

and

$$I_2 = \int_{\mathbb{R}^2} |-2\lambda - i\tau - |\omega|^\alpha|^{-2} \, \mathrm{d}\tau \, \mathrm{d}\omega.$$

Since $\alpha \in (1,2)$, it follows that

$$I_2 = \int_{\mathbb{R}^2} \frac{\mathrm{d}\tau \, \mathrm{d}\omega}{\tau^2 + (2\lambda + |\omega|^\alpha)^2} = \pi \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\lambda + |\omega|^\alpha} := M_3 < \infty.$$

The term I_1 can be estimated as

$$\begin{aligned} I_1 &\leq 2\int_{\mathbb{R}^2} |F[u]|^2(\tau,w)|-\lambda - i\tau|^2 \, \mathrm{d}\tau \, \mathrm{d}\omega + 2\int_{\mathbb{R}^2} |F[u]|^2(\tau,w)\left|-\lambda - \frac{1}{2}\right| w|^\alpha|^2 \, \mathrm{d}\tau \, \mathrm{d}\omega \\ &= 2\int_{\mathbb{R}^2} |F[u_t - \lambda u]|^2(\tau,w) \, \mathrm{d}\tau \, \mathrm{d}\omega + 2\int_{\mathbb{R}^2} |F[\mathcal{L}u - \lambda u]|^2(\tau,w) \, \mathrm{d}\tau \, \mathrm{d}\omega \\ &= 2\|u_t - \lambda u\|_{L_2}^2 + 2\|\mathcal{L}u - \lambda u\|_{L_2}^2. \end{aligned}$$

Thus, we have shown that

$$|u(t,x)|^2 \leq \frac{M_1}{8\pi^4}\left(\|u_t - \lambda u\|_{L_2}^2 + \|\mathcal{L}u - \lambda u\|_{L_2}^2\right)$$

for all $(t,x) \in \mathbb{R}^2$. Estimate (17) then follows because of (16). ■

The estimates from Lemma 3.1 can be extended in the following way.

Corollary 3.2: *Let $f \in L_2(\mathbb{R}^2)$ and λ be any value satisfying the inequality (8). Then,*

(a) *any solution $u \in H(\mathbb{R}^2)$ of Equation (7) satisfies the estimate*

$$\|u\|_H \leq M_1\|f\|_{L_2}, \tag{25}$$

and

(b) *any solution $u \in H(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ of Equation (7) satisfies the estimate*

$$\sup_{(t,x)\in\mathbb{R}^2} |u(t,x)| \leq M_2\|f\|_{L_2}, \tag{26}$$

where the values of M_1 and M_2 depend on ν, μ, K, and α only.

Proof: (a) Since the space $C_c^\infty(\mathbb{R}^2)$ is dense in $H(\mathbb{R}^2)$, there is a sequence of functions $u^n \in C_c^\infty(\mathbb{R}^2)$, $n = 1, 2, \dots$ such that $\|u^n - u\|_H \to 0$ as $n \to \infty$, implying also that $\|u^n - u\|_{L_2}$ as $n \to \infty$.

For $n = 1, 2, \dots$, define

$$f^n := -u_t^n - |\bar{b}|^\alpha \mathcal{L}u^n - \bar{a}u_x^n + \lambda(1 + |\bar{b}|^\alpha)u^n.$$

It can easily be seen that $f^n \in L_2(\mathbb{R}^2)$. Moreover, $\|f^n - f\|_{L_2}$ as $n \to \infty$. The above expression means that, for any fixed $n = 1, 2, \dots$, the function u^n is a solution of Equation (7) with the function f^n. Lemma 3.1 then implies that

$$\|u^n\|_H \leq M_1 \|f^n\|_{L_2}, \tag{27}$$

where the constant M_1 depends on μ, ν, K, and α only. Letting $n \to \infty$ in (27), we obtain

$$\|u\|_H \leq M_1 \|f\|_{L_2},$$

proving estimate (25).

(b) Let $B_N := \{(t,x) \in \mathbb{R}^2 | t^2 + x^2 \leq N^2\}$ and $u^N := uh_N$ for $N = 1, 2, \ldots$, where h_N is a sequence of functions infinitely often differentiable and vanishing outside of B_N, converging increasingly pointwise to 1. It is then clear that $u^N \in C_c^\infty(\mathbb{R}^2), N = 1, 2, \ldots$, and that u^N converges to u as $N \to \infty$ pointwise. We also define

$$u_t^N := \partial_t(u^N), \quad u_x^N := \partial_x(u^N), \quad \mathcal{L}u^N := \mathcal{L}(uh_N)$$

and set

$$f^N := -u_t^N - |\bar{b}|^\alpha \mathcal{L}u^N - \bar{a}u_x^N + \lambda(1 + |\bar{b}|^\alpha)u^N, \quad N = 1, 2, \ldots \tag{28}$$

We observe further that, for all $N = 1, 2, \ldots$,

$$(u^N - u)^2 \leq 2u^2, \quad (u_t^N - u_t)^2 \leq 2u_t^2, \quad (u_x^N - u_x)^2 \leq 2u_x^2, \quad (\mathcal{L}u^N - \mathcal{L}u)^2 \leq 2(\mathcal{L}u)^2,$$

and, since $u \in H(\mathbb{R}^2)$, by Lebesgue's dominated convergence theorem,

$$\|u^n - u\|_{L_2} \to 0, \quad \|u_t^N - u_t\|_{L_2} \to 0, \quad \|u_x^N - u_x\|_{L_2} \to 0, \quad \|\mathcal{L}u^N - \mathcal{L}u\|_{L_2} \to 0,$$

implying $\|f^N - f\|_{L_2} \to 0$ as $N \to \infty$. It is also clear that $f^N \in L_2(\mathbb{R}^2), N = 1, 2, \ldots$.

The relation (28) means that u^N is a solution of Equation (7) with $f = f^N$, so that we obtain by Lemma 3.1

$$|u^N(t,x)| \leq M_1 \|f^N\|_{L_2},$$

holding true for all $(t,x) \in \mathbb{R}^2$. By letting $N \to \infty$ in the above inequality, we arrive at the estimate (26). ∎

4. Some integral estimates

Now, using the analytic estimates from the previous section, we will derive the corresponding integral estimates of the Krylov type for the solutions of stochastic Equations (1) and (6).

First, we choose a non-negative function $\psi(t,x) \in C_c^\infty(\mathbb{R}^2)$ with $\psi(t,x) = 0$ for all (t,x), such that $|t| + |x| \geq 1$ and $\int_{\mathbb{R}^2} \psi(t,x)\, dt\, dx = 1$. For $\varepsilon > 0$, let

$$\psi^{(\varepsilon)}(t,x) = \frac{1}{\varepsilon^2} \psi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Clearly, $\psi^{(\varepsilon)} \in C_c^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} \psi^{(\varepsilon)}(s,x)\, ds\, dx = 1$.

For any function $u \in H(\mathbb{R}^2)$, we define $u^{(\varepsilon)} := u \star \psi^{(\varepsilon)}$ to be the convolution of u with $\psi^{(\varepsilon)}$, i.e.

$$u^{(\varepsilon)}(t,x) = \int_{\mathbb{R}^2} u(s,y)\psi^{(\varepsilon)}(t-s,x-y)\,\mathrm{d}s\,\mathrm{d}y.$$

Since $\|u \star \psi^{(\varepsilon)}\|_{L_2} \le \|u\|_{L_2}\|\psi^{(\varepsilon)}\|_{L_1}$ (see, e.g. Lemma I.8.1, in [8]), it follows that $u^{(\varepsilon)} \in L_2(\mathbb{R}^2)$. Obviously, $u^{(\varepsilon)} \in C^\infty(\mathbb{R}^2)$, and $u^{(\varepsilon)} \to u$ as $\varepsilon \to 0$ a.e. in $\mathbb{R}^2$ and in $L_2(\mathbb{R}^2)$. We also define

$$u_t^{(\varepsilon)} := u_t \star \psi^{(\varepsilon)}, \quad u_x^{(\varepsilon)} := u_x \star \psi^{(\varepsilon)} \tag{29}$$

and note that (see, e.g.[8], Lemma I.8.2)

$$u_t^{(\varepsilon)} = u \star \partial_t\left(\psi^{(\varepsilon)}\right) = \partial_t\left(u^{(\varepsilon)}\right) \quad \text{and} \quad u_x^{(\varepsilon)} = u \star \partial_x\left(\psi^{(\varepsilon)}\right) = \partial_x\left(u^{(\varepsilon)}\right).$$

Moreover, it can be verified directly that, for all $\varepsilon > 0$,

$$\mathcal{L}u^{(\varepsilon)} = (\mathcal{L}u)^{(\varepsilon)}. \tag{30}$$

Theorem 4.1: *Let X be a solution of Equation (6) where $\alpha \in (1,2)$ and the coefficients a and b satisfy condition (5). Then, for any measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a fixed value of λ satisfying the condition (8), it holds that*

$$\mathbf{E}\int_0^\infty e^{-\lambda\phi_s}|f|(s,X_s)\,\mathrm{d}s \le M\|f\|_{L_2}, \tag{31}$$

where $\phi_t = \int_0^t (1+|\bar{b}(s,X_s)|^\alpha)\,\mathrm{d}s$, $t > 0$, and the constant M depends on ν, μ, K, and α only.

Proof: We assume first that $f \in C_c^\infty(\mathbb{R}^2)$. It follows then (see Proposition A.10 in the Appendix) that Equation (7) has a solution $u \in H(\mathbb{R}^2)$.

For $N = 1, 2, \ldots$, define

$$u^N(t,x) := \begin{cases} u(t,x), & \text{if } |u(t,x)| + |u_t(t,x)| + |u_x(t,x)| + |\mathcal{L}u(t,x)| \le N \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

We can see that, for any fixed N, $u^N \in H(\mathbb{R}^2)$, u^N is a bounded function, and $u^N(t,x) \to u(t,x)$ as $N \to \infty$ a.e. in $\mathbb{R}^2$.

Let $u^{N,(\varepsilon)}, u_t^{N,(\varepsilon)}, u_x^{N,(\varepsilon)}$, and $\mathcal{L}u^{N,(\varepsilon)}$ be the corresponding mollified functions for u^N, u_t^N, u_x^N, and $\mathcal{L}u^N$, respectively. Using the above mentioned properties of mollified functions, we can see that, for any $\varepsilon > 0$ and $N = 1, 2, \ldots$, $u^{N,(\varepsilon)} \in H(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$.

For any $\varepsilon > 0$ and $N = 1, 2, ..,$ we also define

$$f^{N,(\varepsilon)} := -u_t^{N,(\varepsilon)} - |\bar{b}|^\alpha \mathcal{L}u^{N,(\varepsilon)} - \bar{a}u_x^{N,(\varepsilon)} + \lambda(1 + |\bar{b}|^\alpha)u^{N,(\varepsilon)}$$

so that the function $u^{N,(\varepsilon)}$ solves the equation

$$u_t^{N,(\varepsilon)} + |\bar{b}|^\alpha \mathcal{L}u^{N,(\varepsilon)} + \bar{a}u_x^{N,(\varepsilon)} - \lambda(1 + |\bar{b}|^\alpha)u^{N,(\varepsilon)} + f^{N,(\varepsilon)} = 0. \tag{33}$$

Applying Lemma I.8.1 in [8], we see that

$$\|u_t^{N,(\varepsilon)}\|_{L_2} \leq \|u_t\|_{L_2}, \|u_x^{N,(\varepsilon)}\|_{L_2} \leq \|u_x\|_{L_2}, \|\mathcal{L}u^{N,(\varepsilon)}\|_{L_2} \leq \|\mathcal{L}u\|_{L_2} \tag{34}$$

for all $\varepsilon > 0$ and $N = 1, 2, \ldots$. By Lebesgue's dominated convergence theorem, it follows then from (33) that $\|f^{N,(\varepsilon)} - f^N\|_{L_2} \to 0$ as $\varepsilon \to 0$, where

$$f^N = -u_t^N - |\bar{b}|^\alpha \mathcal{L}u^N - \bar{a}u_x^N + \lambda(1 + |\bar{b}|^\alpha)u^N \text{ a.e. in } \mathbb{R}^2. \tag{35}$$

Applying Itó's formula to the process $u^{N,(\varepsilon)}(t, X_t)e^{-\lambda\phi_t}, t \geq 0$, (see, e.g.[10], Proposition 2.1) and using Equation (33), we obtain

$$\begin{aligned}
&\mathbf{E}u^{N,(\varepsilon)}(t, X_t)e^{-\lambda\phi_t} - u^{N,(\varepsilon)}(0, x_0) \\
&= \mathbf{E}\int_0^t e^{-\lambda\phi_s}\Big\{u_t^{N,(\varepsilon)}(s, X_s) + |\bar{b}(s, X_s)|^\alpha \mathcal{L}u^{N,(\varepsilon)}(s, X_s) \\
&\quad + \bar{a}(s, X_s)u_x^{N,(\varepsilon)}(s, X_s) - \lambda(1 + |\bar{b}|^\alpha(s, X_s))u^{N,(\varepsilon)}(s, X_s)\Big\}\, \mathrm{d}s \\
&= -\mathbf{E}\int_0^t e^{-\lambda\phi_s} f^{N,(\varepsilon)}(s, X_s)\, \mathrm{d}s
\end{aligned}$$

which yields

$$\mathbf{E}\int_0^t e^{-\lambda\phi_s} f^{N,(\varepsilon)}(s, X_s)\, \mathrm{d}s \leq |u^{N,(\varepsilon)}(0, x_0)| + \mathbf{E}|u^{N,(\varepsilon)}(t, X_t)| \leq 2 \sup_{(s,x)\in\mathbb{R}^2} |u^{N,(\varepsilon)}(s, x)|.$$

Using Corollary 3.2, we obtain

$$\mathbf{E}\int_0^t e^{-\lambda\phi_s} f^{N,(\varepsilon)}(s, X_s)\, \mathrm{d}s \leq M_2\|f^{N,(\varepsilon)}\|_{L_2},$$

and relations (34) together with Lemma 3.1 imply further that

$$\mathbf{E}\int_0^t e^{-\lambda\phi_s} f^{N,(\varepsilon)}(s, X_s)\, \mathrm{d}s \leq M_3\|f\|_{L_2}, \tag{36}$$

where the constants M_2 and M_3 depend on μ, ν, K,and α only.

Using Lebesgue's dominated convergence theorem and (32), we let $\varepsilon \to 0$ in (36) to obtain

$$\mathbf{E}\int_0^t e^{-\lambda\phi_s} f^N(s, X_s)\, \mathrm{d}s \leq M_3\|f\|_{L_2}. \tag{37}$$

Finally, we notice that $u^N = 0$ implies $f^N = 0$, and if $u^N \neq 0$, then it follows that $u^N = u, u_x^N = u_x, u_t^N = u_t$, and $\mathcal{L}u^N = \mathcal{L}u$, so that $f^N = f$ a.e. in $\mathbb{R}^2$. It implies that $|f^N - f| \leq$

$|f|$, and since $|f|$ is a bounded function, we can apply Lebesgue's dominated convergence theorem once again by letting $N \to \infty$ in (37), yielding

$$\mathbf{E} \int_0^t e^{-\lambda \phi_s} f(s, X_s)\, \mathrm{d}s \leq M_3 \|f\|_{L_2}. \tag{38}$$

Now, let $\delta > 0$ and $f_\delta(s,x) := e^{-\delta(s+x)} f(s,x)$, $(s,x) \in \mathbb{R}^2$. Since, for any $f \in C_c^\infty \mathbb{R}^2)$, the function f_δ also belongs to $C_c^\infty(\mathbb{R}^2)$, we can conclude that

$$\mathbf{E} \int_0^t e^{-\lambda \phi_s} e^{-\delta(s+X_s)} f(s, X_s)\, \mathrm{d}s \leq M_2 \Big(\int_{\mathbb{R}^2} e^{-2\delta(s+x)} f^2(s,x)\, \mathrm{d}s\, \mathrm{d}x \Big)^{1/2} \tag{39}$$

for any $t > 0$ and all $f \in C_c^\infty(\mathbb{R}^2)$.

Let $\mathcal{H}$ be the system of all bounded measurable functions f such that (39) holds. Then $\mathcal{H}$ is closed under uniform convergence and under monotone convergence of uniformly bounded sequences. Indeed, if (f^n) is a sequence of such type converging to f, then f^n converges to f pointwise, and, for some $C > 0$, we have $|f^n| \leq C$. Inserting f^n in (39) and applying Lebesgue's dominated convergence theorem on both sides of (39), we get (39) for f. We also note that $\mathcal{A} := C_c^\infty(\mathbb{R}^2)$ is an algebra of functions which generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^2)$. Obviously, there exists $f^n \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq f^n \leq 1$ and $f^n \uparrow 1$ pointwise. Consequently, the assumptions of the *Monotone Class Theorem* (see [3], chapter I, (22.2)) are satisfied. Therefore, we can conclude that (39) holds for all bounded and measurable functions f.

In the next step, we fix an arbitrary bounded measurable function f in (39) and let $\delta \downarrow 0$. The left-hand side converges to the left-hand side of (38) in view of Lebesgue's dominated convergence theorem. The right-hand side of (39) converges to the right-hand side of (38) by monotone convergence. As a result, (38) holds true for every bounded measurable function. Hence

$$\mathbf{E} \int_0^t e^{-\lambda \phi_s} |f|(s, X_s)\, \mathrm{d}s \leq M_2 \|f\|_{L_2} \tag{40}$$

for every bounded measurable function f.

In the last step, let f be an arbitrary measurable function and put $f^n := (f \vee (-n)) \wedge n$, $n \geq 1$. Obviously, we have $|f^n| \uparrow |f|$ and $(f^n)^2 \uparrow f^2$ as $n \to \infty$ pointwise. From inequality (40) being true for f^n, it follows by monotone convergence that (40) holds for f, too.

By Fatou's lemma, in (40) we can let $t \to \infty$, yielding

$$\mathbf{E} \int_0^\infty e^{-\lambda \phi_s} |f|(s, X_s)\, \mathrm{d}s \leq M_2 \|f\|_{L_2},$$

where the constant M_2 depends on μ, ν, K, and α only. Thus, Theorem 4.1 is proven. ■

We can also obtain a local version of estimate (31). For that, for any $t > 0$ and $m \in \mathbb{N}$, we define $\|f\|_{2,m,t} := (\int_0^t \int_{[-m,m]} f^2(s,x)\, \mathrm{d}s\, \mathrm{d}x)^{1/2}$ as the L_2-norm of f on $[0,t] \times [-m,m]$. We also let $\tau_m(X) = \inf\{t \geq 0 : |X_t| > m\}$. Then, applying (31) to the function $\bar{f}(s,x) = f(s,x)\mathbf{1}_{[0,t]\times[-m,m]}(s,x)$, where we set $f(s,x) = 0$ for $s \in (-\infty, 0)$, we obtain the following corollary.

Corollary 4.2: *Let X be a solution of Equation (1) with $\alpha \in (1,2)$ and let assumption (5) be satisfied. Then, for any $t > 0$, $m \in \mathbb{N}$, and any measurable function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, it holds that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} |f|(s, X_s)\,\mathrm{d}s \le M\|f\|_{2,m,t}, \tag{41}$$

where the constant M depends on μ, ν, K, t, α, and m only.

5. Existence of solutions for stochastic equations with measurable coefficients

As an application of the integral estimates derived in the previous section, we prove here the existence of solutions for Equation (1) under assumption (5), where Z is a symmetric stable process of index $\alpha \in (1,2]$.

For $\alpha = 2$, the existence of solutions under (5) is well-known (cf. [7]). Henceforth, we restrict ourselves to the case where $1 < \alpha < 2$.

Theorem 5.1: *Assume that $a, b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are two measurable functions satisfying condition (5) and that $\alpha \in (1,2)$. Then, for any $x_0 \in \mathbb{R}$, there exists a solution of Equation (1).*

Proof: Because of (5), for $n = 1, 2, \ldots$, there are sequences of functions $a_n(t,x)$ and $b_n(t,x)$ such that they are globally Lipschitz continuous, uniformly bounded, and $a_n \to a$, $b_n \to b$ a.e. as $n \to \infty$. Therefore, for any $n = 1, 2, \ldots$, Equation (1) has a unique solution, even a so-called strong solution (see, for example, Theorem 9.1 from chapter 4 in [6]). That is, for any fixed symmetric stable process Z defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, there exists a sequence of processes X^n, $n = 1, 2 \ldots$, such that

$$\mathrm{d}X_t^n = b_n(t, X_{t-}^n)\,\mathrm{d}Z_t + a_n(t, X_t^n)\,\mathrm{d}t, \quad X_0^n = x_0 \in \mathbb{R},\ t \ge 0. \tag{42}$$

Let

$$M_t^n := \int_0^t b_n(s, X_{s-}^n)\,\mathrm{d}Z_s \quad \text{and} \quad Y_t^n := \int_0^t a_n(s, X_s^n)\,\mathrm{d}s$$

so that

$$X^n = x_0 + M^n + Y^n, \quad n \ge 1.$$

As next step, we show that the sequence $H^n = (X^n, M^n, Y^n, Z)$, $n \ge 1$, is tight in the sense of weak convergence in $(\mathbb{D}^4, \mathcal{D}^4)$. Due to the well-known criterion of Aldous ([1]), it suffices to show that

$$\lim_{l\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big[\sup_{0\le s\le t}\|H_s^n\| > l\Big] = 0 \tag{43}$$

for all $t \ge 0$ and

$$\limsup_{n\to\infty}\mathbf{P}\Big[\|H^n_{t\wedge(\tau^n+\delta_n)} - H^n_{t\wedge\tau^n}\| > \varepsilon\Big] = 0 \tag{44}$$

for all $t \ge 0$, $\varepsilon > 0$, for every sequence of $\mathbb{F}$-stopping times τ^n, and for every sequence of real numbers δ_n such that $\delta_n \downarrow 0$. Here $\|\cdot\|$ denotes the Euclidean norm of a vector.

It suffices to verify that the sequence of processes (M^n, Y^n) is tight in $(\mathbb{D}^2, \mathcal{D}^2)$. But this is trivially fulfilled because of the uniform boundness of the coefficients a_n and b_n for all $n \geq 1$.

From the tightness of the sequence $\{H^n\}$, we conclude that there exists a subsequence $\{n_k\}, k = 1, 2, \ldots$ and a process $\bar{H}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ such that H^{n_k} converges weakly (in distribution) to $\bar{H}$ as $k \to \infty$. For simplicity, let $\{n_k\} = \{n\}$.

We now use the well-known principle of Skorokhod (see, e.g. Theorem 2.7 from chapter 1 in [6]) to obtain the convergence of the sequence $\{H^n\}$ a.s. in the following sense: there exist processes $\tilde{H} = (\tilde{X}, \tilde{M}, \tilde{Y}, \tilde{Z})$ and $\tilde{H}^n = (\tilde{X}^n, \tilde{M}^n, \tilde{Y}^n, \tilde{Z}^n), n = 1, 2, \ldots,$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ such that

(1) $\tilde{H}^n \to \tilde{H}$ in $(\mathbb{D}^4, \mathcal{D}^4)$ as $n \to \infty$ $\tilde{\mathbf{P}}$-a.s., and
(2) $\tilde{H}^n = H^n$ in distribution for all $n = 1, 2, \ldots$.

Using standard measurability arguments ([7], chapter 2), we can easily verify that the processes $\tilde{Z}^n$ and $\tilde{Z}$ are symmetric stable processes of index α with respect to the augmented filtrations $\tilde{\mathbb{F}}^n$ and $\tilde{\mathbb{F}}$ generated by the processes $\tilde{H}^n$ and $\tilde{H}$, respectively.

Relying on the above properties (1) and (2), and on Equation (42), we obtain (see, e.g. [7], chapter 2) that

$$\tilde{X}_t^n = x_0 + \int_0^t b_n(s, X_{s-}^n)\tilde{Z}_s^n + \int_0^t a_n(s, \tilde{X}_s^n)\, \mathrm{d}s, \quad t \geq 0, \ \tilde{\mathbf{P}}\text{-a.s.}$$

At the same time, from properties (1), (2) and the quasi-left continuity of the the processes $\tilde{X}^n$, it follows that

$$\lim_{n\to\infty} \tilde{X}_t^n = \tilde{X}_t, \quad t \geq 0, \ \tilde{\mathbf{P}}\text{-a.s.} \tag{45}$$

Hence, in order to show that the process $\tilde{X}$ is a solution of the Equation (1), it is enough to prove that there is a subsequence (n_k) of (n) such that, for all $t \geq 0$, it holds that

$$\lim_{k\to\infty} \int_0^t b_{n_k}(s, \tilde{X}_s^{n_k})\, \mathrm{d}\tilde{Z}_s^{n_k} = \int_0^t b(\tilde{X}_s)\, \mathrm{d}\tilde{Z}_s \quad \tilde{\mathbf{P}}\text{-a.s.} \tag{46}$$

and

$$\lim_{k\to\infty} \int_0^t a_{n_k}(s, \tilde{X}_s^{n_k})\, \mathrm{d}s = \int_0^t a(\tilde{X}_s)\, \mathrm{d}s \quad \tilde{\mathbf{P}}\text{-a.s.} \tag{47}$$

Now we remark that from the convergence in probability it follows that there is a subsequence for which the convergence with probability one holds. Therefore, to verify (46) and (47), it suffices to show that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\left[\left|\int_0^t b_n(s, \tilde{X}_s^n)\, \mathrm{d}\tilde{Z}_s^n - \int_0^t b(s, \tilde{X}_s)\, \mathrm{d}\tilde{Z}_s\right| > \varepsilon\right] = 0 \tag{48}$$

and

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\left[\left|\int_0^t a_n(s, \tilde{X}_s^n)\, \mathrm{d}s - \int_0^t a(s, \tilde{X}_s)\, \mathrm{d}s\right| > \varepsilon\right] = 0. \tag{49}$$

We will also need the following result, which can be proven in the same way as Lemma 4.2 in [9].

Lemma 5.2: *Let $\tilde{X}$ be the process as defined above. Then, for any Borel measurable function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and any $t \geq 0$, there exists a sequence $m_k \in (0,\infty), k = 1,2,\ldots$ such that $m_k \uparrow \infty$ as $k \to \infty$, and it holds that*

$$\mathbf{\tilde{E}} \int_0^{t\wedge\tau_{m_k}(\tilde{X})} |f|(s,\tilde{X}_s)\,\mathrm{d}s \leq M\|f\|_{2,m_k,t},$$

where the constant M depends on λ, α, t, and m_k only. Moreover, it holds that

$$\tilde{\mathbf{P}}\Big[\tau_{m_k}(\tilde{X}^n) < t\Big] \to \tilde{\mathbf{P}}\Big[\tau_{m_k}(\tilde{X}) < t\Big] \text{ as } n \to \infty. \tag{50}$$

Without loss of generality, we can assume $\{m_k\} = \{m\}$.
Let us prove (48) and (49). For a fixed $k_1 \in \mathbb{N}$ we have

$$\begin{aligned}
&\tilde{\mathbf{P}}\Big[\Big|\int_0^t b_n(s,\tilde{X}^n_{s-})\,\mathrm{d}\tilde{Z}^n_s - \int_0^t b(s,\tilde{X}_{s-})\,\mathrm{d}\tilde{Z}_s\Big| > \varepsilon\Big] \\
&\leq \tilde{\mathbf{P}}\Big[\Big|\int_0^t b_{k_1}(s,\tilde{X}^n_{s-})\,\mathrm{d}\tilde{Z}^n_s - \int_0^t b_{k_1}(s,\tilde{X}_{s-})\,\mathrm{d}\tilde{Z}_s\Big| > \frac{\varepsilon}{3}\Big] \\
&\quad + \tilde{\mathbf{P}}\Big[\Big|\int_0^{t\wedge\tau_m(\tilde{X}^n)} b_{k_1}(s,\tilde{X}^n_s)\,\mathrm{d}\tilde{Z}^n_s - \int_0^{t\wedge\tau_m(\tilde{X}^n)} b_n(s,\tilde{X}^n_{s-})\,\mathrm{d}\tilde{Z}^n_s\Big| > \frac{\varepsilon}{3}\Big] \\
&\quad + \tilde{\mathbf{P}}\Big[\Big|\int_0^{t\wedge\tau_m(\tilde{X})} b_{k_1}(s,X_s)\,\mathrm{d}\tilde{Z}_s - \int_0^{t\wedge\tau_m(\tilde{X})} b(s,\tilde{X}_{s-})\,\mathrm{d}\tilde{Z}_s\Big| > \frac{\varepsilon}{3}\Big] \\
&\quad + \tilde{\mathbf{P}}\Big[\tau_m(\tilde{X}^n) < t\Big] + \tilde{\mathbf{P}}\Big[\tau_m(\tilde{X}) < t\Big].
\end{aligned}$$

The first term on the right side of the inequality above converges to 0 as $n \to \infty$ by Chebyshev's inequality and Skorokhod's lemma for stable integrals (see [12], Lemma 2.3). To show the convergence to 0 as $n \to \infty$ of the second and third terms we use first Chebyshev's inequality and then Corollary 41 and Lemma 5.2, respectively. We obtain

$$\begin{aligned}
&\tilde{\mathbf{P}}\Big[\Big|\int_0^{t\wedge\tau_m(\tilde{X}^n)} b_{k_1}(s,\tilde{X}^n_s)\,\mathrm{d}\tilde{Z}^n_s - \int_0^{t\wedge\tau_m(\tilde{X}^n)} b_n(s,\tilde{X}^n_{s-})\,\mathrm{d}\tilde{Z}^n_s\Big| > \frac{\varepsilon}{3}\Big] \\
&\quad \leq \frac{3}{\varepsilon}\tilde{\mathbf{E}}\Big|\int_0^{t\wedge\tau_m(s,\tilde{X}^n)} |b_{k_1} - b_n|^\alpha(s,\tilde{X}^n_{s-})\,\mathrm{d}s\Big| \leq \frac{3}{\varepsilon}M\||b_{k_1} - b_n|^\alpha\|_{2,m,t}
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
&\tilde{\mathbf{P}}\Big[\Big|\int_0^{t\wedge\tau_m(\tilde{X})} b_{k_1}(s,X_s)\,\mathrm{d}\tilde{Z}_s - \int_0^{t\wedge\tau_m(\tilde{X})} b(s,\tilde{X}_{s-})\,\mathrm{d}\tilde{Z}_s\Big| > \frac{\varepsilon}{3}\Big] \\
&\quad \leq \frac{3}{\varepsilon}\tilde{\mathbf{E}}\Big|\int_0^{t\wedge\tau_m(\tilde{X})} |b_{k_1} - b|^\alpha(s,\tilde{X}_{s-})\,\mathrm{d}s\Big| \leq \frac{3}{\varepsilon}M\||b_{k_1} - b|^\alpha\|_{2,m,t}
\end{aligned} \tag{52}$$

where the constant M depends on μ, ν, K, m, t, and α only.

It follows from the definition of the sequence b_n that, for any $t > 0$ and $m \in \mathbb{N}$, $|b_{k_1} - b_n|^\alpha \to 0$ by letting first n and then k_1 tend to infinity. Similarly, $|b_{k_1} - b|^\alpha \to 0$ as $k_1 \to \infty$ in the $L_{2,m,t}$-norm. Then, passing to the limit in (51) and (52) first $n \to \infty$ and then $k_1 \to \infty$, we obtain that the right end sides of (51) and (52) converge to 0.

Because of (50), the remaining terms $\tilde{\mathbf{P}}[\tau_m(\tilde{X}^n) < t]$ and $\tilde{\mathbf{P}}[\tau_m(\tilde{X}) < t]$ can be made arbitrarily small by choosing large enough m for all n, due to the fact that the sequence of processes $\tilde{X}^n$ satisfies (43). This verifies (48). The convergence (49) can be verified similarly, so we omit the details.

Thus, we have proven the existence of a process $\tilde{X}$ that solves the Equation (1). ■

Note

1. $C_c^\infty(\mathbb{R}^2)$ defines the class of infinitely differentiable functions with compact support in $\mathbb{R}^2$.

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Appendix

Here we prove the existence of a solution of Equation (7) in the Sobolev space $H(\mathbb{R}^2)$ for any $f \in C_c^\infty(\mathbb{R}^2)$ and with coefficients a and b satisfying condition (5). We use the method of continuity and the method of a priori estimates in a similar way as in [8] for classical elliptic and parabolic equations.

We start with the equation

$$u_t + \mathcal{L}u - \lambda u = f, \tag{A1}$$

where $\lambda > 0$.

To solve (A1) in $H(\mathbb{R}^2)$, we will need several lemmas and a corollary.

Lemma A.3: *Let $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ and $u \in C_c^\infty(\mathbb{R}^2)$ be a solution of (A1). Then, it holds that*

$$\|u_t\|_{L_2}^2 + \lambda^2\|u\|_{L_2}^2 + \|\mathcal{L}u\|_{L_2}^2 \le \|f\|_{L_2}^2. \tag{A2}$$

Proof: Applying the Fourier transform in variables (t, x) to Equation (A1) and using Proposition 2.1, we obtain

$$-i\tau F[u] - (\lambda + |w|^\alpha)F[u] = F[f],$$

or

$$\left(|\tau|^2 + (\lambda + |w|^\alpha)^2\right)|F[u]|^2 = |F[f]|^2,$$

which implies

$$|\tau|^2|F[u]|^2 + \lambda^2|F[u]|^2 + |w|^{2\alpha})|F[u]|^2 \le |F[f]|^2.$$

Integrating the last relation over $\mathbb{R}^2$ and using Plancherel's identity, we obtain (A2). ■

Corollary A.4: *Let $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ and $u \in C_b^\infty(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ be a solution of Equation (A1). Then, for any $\lambda > 0$,*

$$\|u\|_{L_2} \le \frac{1}{\lambda}\|f\|_{L_2}. \tag{A3}$$

Proof: Since $C_c^\infty(\mathbb{R}^2)$ is dense in $L_2(\mathbb{R}^2)$, there is a sequence of functions $u^n \in C_c^\infty(\mathbb{R}^2), n = 1, 2, \ldots$ so that $\|u^n - u\|_{L_2} \to 0$ as $n \to \infty$. Set

$$f^n := -u_t^n - \mathcal{L}u^n + \lambda u^n, \quad n = 1, 2, \ldots.$$

It can easily be seen that, for any $n = 1, 2, \ldots, f^n \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ and u^n solves the equation

$$u_t^n + \mathcal{L}u^n - \lambda u^n = f^n. \tag{A4}$$

Using (A1) and (A4), we obtain that $\|f^n - f\|_{L_2}$ as $n \to \infty$.

Lemma A.3 implies then

$$\|u^n\|_{L_2} \le \frac{1}{\lambda}\|f^n\|_{L_2},$$

and upon letting $n \to \infty$, we arrive at (A3). ■

The next statement is an immediate consequence of Corollary A.4 with $f = 0$.

Lemma A.5: *Let $u \in C_b^\infty(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ be a solution of equation*

$$u_t + \mathcal{L}u - \lambda u = 0.$$

Then $u = 0$ a.e.

Now, we consider the set of functions

$$\mathcal{A} := \{g : g(t,x) = \partial_t u(t,x) + \mathcal{L}u(t,x) - \lambda u(t,x) \text{ for some } u \in C_c^\infty(\mathbb{R}^2)\}.$$

Lemma A.6: *The set $\mathcal{A}$ is dense in $L_2(\mathbb{R}^2)$.*

Proof: It is enough to prove that $\mathcal{A}^\perp = \{0\}$ where $\mathcal{A}^\perp$ is the orthogonal complement of $\mathcal{A}$ in $L_2(\mathbb{R}^2)$. For that, we choose an arbitrary function $h \in L_2(\mathbb{R}^2)$ so that

$$\int_{R^2} h(t,x)\Big(\partial_t + \mathcal{L} - \lambda\Big)u(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0$$

for all $u \in C_c^\infty(\mathbb{R}^2)$. We have to verify that $h = 0$.

The last relation also implies that

$$\int_{R^2} h(t,x)\Big(\partial_t + \mathcal{L} - \lambda\Big)u(\tau - t, y - x)\,\mathrm{d}t\,\mathrm{d}x = 0, \tag{A5}$$

since $u(\tau - \cdot, y - \cdot) \in C_c^\infty(\mathbb{R}^2)$ for all fixed $(\tau, y) \in \mathbb{R}^2$.

Using convolution, (A5) is then written as

$$h \star \frac{\partial}{\partial t}u(\tau,y) + h \star \mathcal{L}u(\tau,y) - \lambda h \star u(\tau,y) = 0. \tag{A6}$$

Clearly,

$$h \star \frac{\partial}{\partial t}u = \frac{\partial}{\partial t}\Big(h \star u\Big). \tag{A7}$$

We also have that

$$\begin{aligned}
&h \star \mathcal{L}u(\tau,y)\\
&= \int_{\mathbb{R}^2} h(t,x)\mathcal{L}u(\tau - t, y - x)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2} h(t,x)\int_{\mathbb{R}}\Big[u(\tau - t, y - x + z) - u(\tau - t, y - x) - \mathbf{1}_{|z|<1}u_x(\tau - t, y - x)z\Big]\frac{\mathrm{d}z}{|z|^{1+\alpha}}\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}(h \star u)(\tau,y)\\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}^2} h(t,x)u(\tau - t, y - x + z)\,\mathrm{d}t\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^2} h(t,x)u(\tau - t, y - x)\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{R}^2} zh(t,x)u_x(\tau - t, y - x)\mathbf{1}_{|z|<1}\,\mathrm{d}t\,\mathrm{d}x\Big)\frac{\mathrm{d}z}{|z|^{1+\alpha}}\\
&= \int_{\mathbb{R}^2} h(t,x)\int_{\mathbb{R}}\Big[u(\tau - t, y - x + z) - u(\tau - t, y - x) - \mathbf{1}_{|z|<1}u_x(\tau - t, y - x)z\Big]\frac{\mathrm{d}z}{|z|^{1+\alpha}}\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

where we used the fact that $(h \star u)_x = h \star u_x$.

Comparing the above relations, we conclude that

$$h \star \mathcal{L}u = \mathcal{L}(h \star u). \tag{A8}$$

Using (A7) and (A8), Equation (A6) becomes

$$\Big(\partial_t + \mathcal{L} - \lambda\Big)h \star u(\tau,y) = 0.$$

We also observe that $h \star u \in C_b^\infty(\mathbb{R}^2)$. Indeed, any derivative of $h \star u$ is equal to a convolution of h with the corresponding derivative of u. The claim then follows from the Cauchy-Schwarz inequality, since h, $\partial_t u$, and $\partial_x u$ are all L_2 functions.

Applying Lemma A.5, we obtain

$$h \star u(\tau, y) = \int_{\mathbb{R}^2} h(t,x)u(\tau - t, y - x)\, \mathrm{d}t\, \mathrm{d}x = 0$$

for all $u \in C_c^\infty(\mathbb{R}^2)$ and a.e. $(\tau, y) \in \mathbb{R}^2$. It follows from the general integration theory that $h = 0$ a.e. in $\mathbb{R}^2$, implying $\|h\|_{L_2} = 0$. ■

Lemma A.7: *Let $\lambda > 0$ and $f \in C_c^\infty(\mathbb{R}^2)$. Then there is a solution $u \in H(\mathbb{R}^2)$ of the Equation (A1).*

Proof: By Lemma A.6, there is a sequence of functions $u^n \in C_c^\infty(\mathbb{R}^2)$ so that

$$\left(u_t^n + \mathcal{L}u^n - \lambda u^n\right) \to f \text{ as } n \to \infty$$

in $L_2(\mathbb{R}^2)$.

Define

$$f^n := \left(u_t^n + \mathcal{L}u^n - \lambda u^n\right), \quad n = 1, 2, \ldots \tag{A9}$$

Using Lemma A.3, we obtain that

$$\|u_t^n - u_t^m\|_{L_2}^2 + \lambda^2\|u^n - u^m\|_{L_2}^2 + \|\mathcal{L}u^n - \mathcal{L}u^m\|_{L_2}^2 \le \|f^n - f^m\|_{L_2}^2$$

for all $n, m = 1, 2, \ldots$

Since (f^n) converges in $L_2(\mathbb{R}^2)$, it is a Cauchy sequence so that $\|f^n - f^m\|_{L_2} \to 0$ as $n, m \to \infty$. This implies that the sequences (u^n), (u_t^n), and $(\mathcal{L}u^n)$ are also Cauchy sequences. Because of the completeness of $L_2(\mathbb{R}^2)$, the following limits exist in $L_2(\mathbb{R}^2)$:

$$v(t,x) := \lim_{n\to\infty} u^n(t,x), \quad \tilde{u}(t,x) := \lim_{n\to\infty} u_t^n(t,x), \quad \hat{u}(t,x) := \lim_{n\to\infty} \mathcal{L}u^n(t,x).$$

Using similar arguments as in [8] (see, e.g. Lemma 3 and Theorem 11 in chapter 1), one can show then that v_t exists and is independent of the choice of defining sequence. Also, if $u^{n,1}$ and $u^{n,2}$ are two defining sequences for u, then we can easily verify that $\lim_{n\to\infty} \mathcal{L}u^{n,1}$ and $\lim_{n\to\infty} \mathcal{L}u^{n,2}$ in $L_2(\mathbb{R}^2)$ coincide, so that we can define the closure of the operator $\mathcal{L}$ on the space $L_2(\mathbb{R}^2)$ as $\hat{\mathcal{L}}u := \lim_{n\to\infty} \mathcal{L}u^n$, where $u \in L_2(\mathbb{R}^2)$, and u^n is a defining sequence for u. For simplicity, we use the same notation $\mathcal{L}$ for the closure operator $\hat{\mathcal{L}}$. In particular, if $u \in C_c^\infty(\mathbb{R}^2)$, then $\hat{\mathcal{L}}u = \mathcal{L}u$.

It follows then from (A9) that

$$v_t + \mathcal{L}v - \lambda v = f \text{ a.e. in } \mathbb{R}^2.$$

Therefore, v is a solution of Equation (A1) in the sense described above, which is often referred to as *a generalized solution in the Sobolev space H*. ■

Remark A.8: For an alternative way to solve Equation (A1), we refer to [4], where it is shown that the solution of (A1) can be written as

$$u(t,x) = -\int_t^\infty e^{-\lambda(s-t)}\, \mathrm{d}s \int_{\mathbb{R}} g(s-t,x,y) f(s,y)\, \mathrm{d}y,$$

where $g(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i(x-y)\xi - \frac{1}{2}t|\xi|^\alpha)\, \mathrm{d}\xi$.

In particular, the authors derive estimates for the kernel function g and its fractional derivatives, which then can be used to solve Equation (A1).

Now, for $\lambda > 0$ and $\alpha \in (1,2)$, we consider the operator

$$L := \partial_t + |\bar{b}|^\alpha \mathcal{L} + \bar{a}\partial_x - \lambda(1 + |\bar{b}|^\alpha),$$

where the real-valued functions a, b satisfy assumption (5) and $\bar{a}, \bar{b}$ are their extensions as defined in Section 1.

For any $s \in [0,1]$, we set

$$L_s := (1-s)(\partial_t + \mathcal{L} - \lambda) + sL.$$

The following result is an analog of Theorem 1.4.4 from [8]. The proof is entirely based on general functional analysis and we refer for details to [8].

Proposition A.9: *Assume that there are constants $\lambda > 0$ and $M \in (0,\infty)$ such that for any $u \in C_c^\infty(\mathbb{R}^2)$ and $s \in [0,1]$ it holds that*

$$\|u\|_H \le M\|L_s u\|_{L_2}. \tag{A10}$$

Then, for any $f \in C_c^\infty(\mathbb{R}^2)$, there is a function $u \in H(\mathbb{R}^2)$ satisfying $Lu = f$.

Condition (A10) can be reformulated as follows: for any $u \in H(\mathbb{R}^2)$ satisfying the equation $L_s u = f$, it holds that

$$\|u\|_H \le M\|f\|_{L_2}. \tag{A11}$$

Estimate (A11) is called *an a priori estimate* for the equation $L_s u = f$, since we do not yet know the existence of such a solution.

Proposition A.10: *For any function $f \in C_c^\infty(\mathbb{R}^2)$ and a fixed value of $\lambda > 0$ satisfying condition (8), there is a solution $u \in H(\mathbb{R}^2)$ of the equation $Lu = f$.*

Proof: Let us first prove the statement for $\bar{a} = 0$.

It follows from Lemma 3.1 that, for any $u \in C_c^2(\mathbb{R}^2)$ and λ satisfying (8), it holds that

$$\|u_t\|_{L_2}^2 + \lambda^2\|u\|_{L_2}^2 + \|\mathcal{L}u\|_{L_2}^2 \le M\|u_t + |\bar{b}|^\alpha \mathcal{L}u - \lambda(1+|\bar{b}|^\alpha)u\|_{L_2}^2, \tag{A12}$$

where the constant M depends on ν and μ only.

For $s \in [0,1]$, we consider

$$\tilde{L}_s u := (1-s)(u_t + \mathcal{L}u - 2\lambda u) + s\Big(u_t + |\bar{b}|^\alpha \mathcal{L}u - \lambda(1+|\bar{b}|^\alpha)u\Big).$$

It can easily be seen that

$$\begin{aligned}\tilde{L}_s u &= u_t + [1 - s + s|\bar{b}|^\alpha]\mathcal{L}u - \lambda[1 + 1 - s + s|\bar{b}|^\alpha]u\\ &= u_t + \sigma(s)\mathcal{L}u - \lambda[1+\sigma(s)]u,\end{aligned}$$

where

$$\sigma(s) = 1 - s + s|\bar{b}|^\alpha.$$

Because of Lemma A.7, the equation $u_t + \mathcal{L}u - 2\lambda u = f$ has a solution $u \in H(\mathbb{R}^2)$ for any λ satisfying (8) and $f \in C_c^\infty(\mathbb{R}^2)$. By Proposition A.9, the claim is then proved if, for any $s \in [0,1]$ and any $u \in C_c^\infty(\mathbb{R}^2)$, it follows that

$$\|u\|_H \le M\|\tilde{L}_s u\|_{L_2}.$$

The latter, however, follows from (A12) if we replace $|\bar{b}|^\alpha$ by $\sigma(s)$ and note that, for any $s \in [0,1]$, it holds that

$$0 < \min\{1, \mu^\alpha\} \le \sigma(s) \le \max\{1, \nu^\alpha\}$$

since $\sigma(s)$ is a linear function in s.

To prove the general case, we consider, for $s \in [0,1]$, the operator

$$\begin{aligned}L_s u &= (1-s)\Big(u_t + |\bar{b}|^\alpha \mathcal{L}u - \lambda(1+|\bar{b}|^\alpha)u\Big) + sLu\\ &= u_t + |\bar{b}|^\alpha \mathcal{L}u - \lambda(1+|\bar{b}|^\alpha)u + s\bar{a}u_x.\end{aligned}$$

Using (A12), we obtain that, for any $u \in C_c^\infty(\mathbb{R}^2)$ and λ satisfying (8), it holds that

$$\|u_t\|_{L_2} + \lambda\|u\|_{L_2} + \|\mathcal{L}u\|_{L_2} \le M_1\|L_s\|_{L_2} + M_2\|u_x\|_{L_2}, \tag{A13}$$

where the constants M_1 and M_2 depend on the bounds of the coefficients $\bar{a}$ and $\bar{b}$.

It can be easily seen that, for any fixed $1 < \alpha < 2$, there exists λ_0 satisfying (8) so that

$$M_2|\omega|^2 \leq \frac{1}{2}(\lambda_0 + |\omega|^\alpha)^2, \quad \omega \in \mathbb{R}.$$

It follows then that

$$M_2\|u_x\|_{L_2} \leq \frac{1}{2}\|\mathcal{L}u\|_{L_2} + \frac{\lambda_0}{2}\|u\|_{L_2},$$

and by (A13) we conclude that

$$\|u_t\|_{L_2} + \left(\lambda - \frac{\lambda_0}{2}\right)\|u\|_{L_2} + \frac{1}{2}\|\mathcal{L}u\|_{L_2} \leq M_1\|L_s u\|_{L_2}.$$

The last relation implies the a priori estimate

$$\|u\|_H \leq M\|L_s u\|_{L_2}$$

for $\lambda > \lambda_0/2$ with M depending on the bounds of $\bar{a}$ and $\bar{b}$. The latter, in turn, implies the existence of a solution $u \in H(\mathbb{R}^2)$ of the equation $Lu = f$ for any $f \in C_c^\infty(\mathbb{R}^2)$ because of Proposition A.9. ■