# Market Timing and Debt Maturity: Liquidity Risk vs. Default Risk*

Rody Manuelli
Washington University in St. Louis
Federal Reserve Bank of Saint Louis

Juan M. Sánchez
Federal Reserve Bank of Saint Louis

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**Abstract**

We study the endogenous determination of debt maturity in an environment where the financial structure of the firm is determined by market timing. Firms have access to a bond with a flexible structure. The optimal bond maturity balances liquidity risk and default risk. We find that market timing implies that firms with poor prospects and firms in more unstable industries will choose shorter maturities even if it is feasible to issue longer debt. The model also offers predictions on how asset maturity, asset salability, and leverage influence maturity. Even though our model is stylized, predictions are roughly consistent with the evidence.

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# 1 Introduction

Baker and Wurgler (2002) showed that there are significant fluctuations in the cost of issuing debt relative to issuing equity, and argued that these fluctuations provide an explanation for the capital structure of firms. Using different data sources and methods, Jermann and Quadrini (2012) documents that equity payouts are negatively correlated with debt repurchases, suggesting some substitutability between equity and debt financing. Motivated by these facts, we assume there are two phases for financing: a liquid phase, in which firms prefer to issue equity; and an illiquid phase, in which firms prefer to issue bonds. The rest of the paper studies what are the implications of these fluctuations for the choice of corporate debt maturity with risk of default.

The model delivers an optimal maturity that depends on aggregate factors (e.g. duration of the illiquid phase) and firms' characteristics (e.g. earnings growth and volatility). Corporate debt has risk of default because firms earnings are stochastic. At any point, firms compare the value of repayment and default and choose the best option. Market timing is captured with a sharp assumption: financial markets move stochastically between liquid and illiquid times. During the *illiquid* times, the cost of issuing equity is prohibitively expensive and firms must finance investment and rollover debt by issuing a bond. The choice of maturity at the time of issuing a bond is the central issue studied in this paper. A firm chooses the type of debt from a general menu of non-contingent securities that we view as bonds. They are completely summarized by three parameters: the coupon rate, $b$, that must be paid in every period, the face value (or the value that is due at maturity), $K$, and a Poisson parameter, $\eta$, that determines the stochastic maturity of the bond. The expected maturity of the bond is then given by $1/\eta$. During *liquid* times the model is very simple. Firms can issue equity and they will do so to repay maturing debt, unless they prefer to default.

To highlight some of the economic forces that underlie the choice of maturity we first discuss a simple version of the model that rules out refinancing.[1] In this setting, we show that the market value of the debt relative to its safe value depends on two risk prices (or discounts). Liquidity risk captures the risk that the debt matures during the illiquid regime,

[1] Since issuing a bond is costly, this is an extreme case with a very large cost for refinancing.

while default risk measures the cost of strategic default. We next demonstrate how the state of the firm and the economic environment affect these two prices of risk, and find that disentangling these effects provides intuition about the optimal maturity choice.

We tackle the analysis of the general model in the context of a quantitative exercise. We first study the more common form of debt: pure discount bonds. Thus, the two elements of the debt are its average duration and its face value. The structure of the debt depends both on the value of the unlevered firm[2] and on features of the project/technology. Firms whose value is low—and are hence more likely to default for strategic or solvency reasons—choose *shorter* bonds. As the unlevered value of the firm increases, the risk of strategic default decreases and the optimal bond is chosen so as to decrease the risk of default associated with the bond maturing in the low growth regime. This is accomplished by increasing expected maturity of the bond at the cost of a higher face value and, hence, of a higher premium paid for strategic default. An implication is the firms that need to refinance their debt at the time when prospects are poor will choose shorter maturities as it balances the two types of risk. Outside observers without a full understanding of the problem solved by the firm might be tempted to reverse the direction of causality and conclude that the firm made a "mistake" issuing short debt. This, in turn, would exacerbate the chances of default.

The (fixed) properties of the technology have an impact on the choice of maturity: firms that operate in more unstable environments choose shorter maturities and we find that the expected relationship between maturity and output is flatter the higher the uncertainty about growth rates. Thus, the model not only implies that firms in high uncertainty environments borrow using more short-term debt than similar firms in a low uncertainty environment, but also that the cross-sectional dispersion of expected maturities is smaller in the high uncertainty case. Our results show that the degree of salability of the firm's assets (or, alternatively the cost of fire sales) also influences the choice of the optimal financing structure. We find that higher post-default values of the assets result in longer expected maturities. This is intuitive: A higher recovery rate, conditional on default, reduces the cost of strategic default and hence its shadow price. The firm then chooses to reduce the higher price risk—the risk of illiquidity driven default—by choosing a bond with longer duration.

[2]This is the expected present discounted value of the firm if it was 100% equity financed.

For the examples that we study, higher levered firms (i.e. firms with higher financing needs) choose shorter maturities while, at the same time, selling bonds with similar face values. Thus, unlike other features of the environment, changes in leverage have a large impact on expected maturity and a small impact on the face value of the optimal bond. Not surprisingly, we find that the optimal choice of debt structure cannot be summarized as the minimization of a simple measure of cost like the excess yield.

We then explore how sensitive our results are to the assumption that the firm is restricted to issuing pure discount bonds. To do this we allow for the possibility of issuing debt that promises a coupon payment as well as a face value at maturity. We present results when we restrict the choice of maturities to a fixed set. We find that when the firm is at a corner —i.e. issuing the debt with the highest possible coupon— the choice of maturity is roughly similar to the pure discount bond. However, as the (unlevered) value of the firm increases the optimal choice of debt structure becomes less monotonic. In our quantitative model we find that there is a trade-off between smaller coupons —which reduces the size of strategic default before the bond matures— and expected maturity: Higher value firms issue bonds with lower coupons and shorter maturities. Thus, at the high end of the value distribution we find firms issuing relatively short debt.

The endogenous determination of the maturity structure of debt is a topic that interests both financial and macro economists. In the case of business firms—the case mostly analyzed by financial economists—early work by Diamond (1991) and by Leland and Toft (1996) has been followed by substantial work on the optimal structure of debt. There are important advantages of our flexible specification of the structure of debt over the constant maturity approach pioneered by Leland and Toft (1996) and then adopted by He and Xiong (2012), Diamond and He (2014), and He and Milbradt (2014). First, it allows us to study debt that can be either front or back-loaded, and provides a useful setting to capture the trade-off between expected maturity and face value of a bond. Second, at refinancing time, the firm is completely free to change the structure of the debt. This is essential to understanding how changes in the prospects of the firm get reflected in changes in maturity.[3]

[3] In a recent paper, Chen et al. (2012) use a shortcut to allow for a change in maturity when the aggregate state changes within the constant maturity model.

We purposely abstract away from the strategic aspects associated with issuing short and long-term debt simultaneously since there is no risk of debt dilution. This is extensively studied in the sovereign debt literature by Arellano and Ramanarayanan (2012), Hatchondo et al. (2016), Sánchez et al. (2018), and Aguiar et al. (2019). Moreover, we also ignore how the liquidity of the secondary market influences the choice of maturity which has been the focus of the recent literature (see, for example, Kozlowski, forthcoming). Our emphasis is on the non-strategic determinants of debt maturity, the role played by technology, and the fact that firms are timing the market to issue equity.

In the macro literature, Lucas and Stokey (1983) highlighted the role that debt maturity has in supporting an optimal policy. Angeletos (2009) and Buera and Nicolini (2004) study the case of non-contingent debt while Shin (2007) allows for a maturity structure that changes depending on the state of the economy. In all cases, there is no default in equilibrium.

## 1.1 Review of Empirical Findings

In this section we report some results on the variation, along several dimensions, of the equilibrium debt structure chosen by firms. Since the literature is very broad, we provide here a short description of the studies that emphasize the factors that we highlight in our setting.[4] In Appendix A, we reproduce some of those findings using non-financial and non-regulated firms in Standard & Poor's Compustat.[5]

We find that the evidence on the relationship between firm value and debt maturity is sensitive to details on how these concepts are measured. Barclay and Smith (1995) use Compustat data to study debt maturity. They restrict their analysis to only firms in the industrial corporate sector. Debt maturity is measured as the share of the firm's total debt with a maturity longer than 3 years, which is a very common choice given the information provided by Compustat. They find a negative relationship between R&D investment / firm value and debt maturity. This result seems to contradict our theory, which predicts that

[4] Some articles that we do not include here empirically analyze debt maturity in the context of the capital structure of the firm. See, for instance, Titman and Wessels (1988).

[5] Most of the empirical literature on determinants of debt maturity uses data on publicly traded companies. Ideally, one would like to analyze data on all firms but available balance sheet data on non-traded firms is very limited.

firms with higher potential growth (as measured by the expected growth rate of earnings) would prefer to issue longer term debt. Guedes and Opler (1996) reproduce some of the findings in Barclay and Smith (1995) using an alternative dataset with information about debt issuances. However, they find a positive relationship between R&D investment-to-sales ratio and debt maturity. Our findings in Appendix A, using a broader sample of Compustat firms, are in line with Guedes and Opler (1996).

Stohs and Mauer (1996) also use a broad sample of Compustat firms to analyze the determinants of debt maturity. They find that less risky firms use longer-term debt. A similar finding is reported by Okzan (2002) for UK firms. In Appendix A, we also find that firms in more volatile environments tend to issue shorter-term debt. They argue that this may be because riskier firms need to re-balance their capital structure to moderate expected bankruptcy costs. Our model provides an alternative explanation for this finding.

Guedes and Opler (1996), Stohs and Mauer (1996) and Okzan (2002) also show that firms with longer-term asset maturities use longer-term debt. This relationship is expected from the idea of "matching principle," where firms would like to match the maturity of their assets to the maturity of their debt in order to avoid defaulting when the debt matures earlier. In Appendix A, we also find that firms with a higher proportion of assets maturing at a long-term period also issue debts with longer-maturity, though this coefficient is not statistically significant.

Demirgüç-Kunt and Maksimovic (1999) find that firms with higher asset salability, thus higher post-default value of assets, are associated with longer maturity. This result is also consistent with the findings of Benmelech (2009), who reports that American railroads which used rolling stock that could be more easily redeployed (higher resale value) tended to issue longer term bonds. Our results in Appendix A are also in line with this finding.

Barclay et al. (2003) find that leverage is negatively associated with debt maturity. In contrast, Johnson (2003) finds a positive relationship between maturity and leverage, and argues that firms with high leverage choose longer term debt to avoid liquidation. This effect is probably more important for firms with a very high risk of default. Our results in Appendix A using Compustat firms suggest that firm with higher leverage (actually, with more reliance on external financing) prefer shorter term debt. In our theory, firms with higher leverage would choose shorter debt maturity.

Section 3 presents the model and section 4 describes the results for the case of extreme illiquidity. Section 5 presents our quantitative results. Section 6 contains some concluding remarks.

## 2 Example: Illiquidity Risk vs. Strategic Risk

In order to highlight the interplay between technological parameters and the choice of financing we describe some results in a special case of the general model as it provides insights into the factors that influence the choice of maturity.

There is a risk-neutral firm that has to incur an irreversible cost to implement an investment project. The returns of the project are completely described by a few parameters that determine the associated stochastic processes for output (or earnings). Net earnings are given by a stochastic process $x_t$ such that

$$dx_t = \mu x_t dt + \sigma x_t dW_t,$$

where $W_t$ is a Wiener process or Brownian motion, and $\mu$ ("the percentage drift") and $\sigma$ ("the percentage volatility") are constants. Thus, the the technology is described by $(\mu, \sigma)$.

As in the general model, there are two phases. By assumption, in the **illiquid regime**, the firm has no access to equity markets and must finance itself with the bond market. The amount of time spent in this regime is random and distributed exponentially with parameter $\upsilon$. In this regime, the firm has access to a risk-neutral credit market that will price any debt issued by the firm using the same discount rate, $r$. Thus, none of our results are driven by difference in discount factor nor difference in the curvature of direct payoffs. We study a bond market wherein the set of feasible debt instruments is completely described by three parameters $(b, K, \eta)$, where $b$ is the coupon rate, $K$ is the payment due at maturity (which does not necessarily coincide with the initial value of the debt) and $\eta$ is the parameter of a Poisson process that determines the maturity of the bond. Thus, maturity is stochastic and the maturity time is denoted $T_\eta$. The expected maturity of a bond is $1/\eta$.[6] Given that the length of the illiquid phase is uncertain, even if maturity was not stochastic by choosing higher maturity firms would be choosing a lower probability that the debt matures in the illiquid phase.

The first simplifying assumption in this section is that after issuing a bond in the illiquid phase the firm cannot rollover debt. This is a magnified version of our risk of default due to illiquidity. With this assumption, note that the firm has clear incentives to issue a long bond to mitigate the risk of defaulting just because the bond matures before the switch to the liquid phase.[7]. The second difference with the general model is that during the illiquid phase there is a low and constant value of earnings $z$ while $x$ are potential earnings that fluctuate but they are realized only after the switch to the liquid phase. This assumption is made such that there is not "strategic default" during the illiquid phase.

We make these two assumptions to magnify the differences between defaults due to liquidity, which here will occur when the bond matures in the illiquid phase, and defaults due to strategic reasons, which here may occur only if the bond matures after the switch to the liquid phase.

## 2.1 Two reasons for default

First, note that during the illiquid regime the relevant case is that the firm issues a bond such that it has enough (safe) earnings to pay the coupon $b$ (formally, $z \geq b$). Otherwise, the firm would be force to default immediately and, consequently, the value of the bond would be zero. Thus, for a bond satisfying $z \geq b$, default occurs if the debt matures in this regime, independently of the level of potential earnings $x$. This is what we refer to as illiquidity risk.

More things may happen during the liquid regime. The firm may default before maturity if the value of $x$ reaches $x^*$. Or the firm may default at maturity if $x < \bar{x}(K)$. Depending

[6]We only consider bonds which make payments that are not contingent on the state of the firm. In particular, we disallow callable bonds.

[7]In the general model, the firm will have access to issuing a new bond to rollover the maturing debt in the illiquid phase. However, there risk of default will be higher than if the bond matures in the illiquid phase because (i) there will be a cost of issuing a new bond and (ii) given the firm's current $x$ at maturity the lenders may ask for a high interest rate because the project became to risky (it may even be that they are not willing to refinance it)

on the structure of the bond, that is the values of $(b, \eta, K)$, the firm will choose to default strategically. For a bond with relatively small value of $K$, default may occur only before maturity. This corresponds to $x^* > \bar{x}(K)$. For a bond with relatively large value of $K$, the firm may either default before or at the time that the debt matures. This corresponds to $x^* < \bar{x}(K)$. The previous two scenarios correspond to what we label strategic risk.

Finally, note that to simplify the algebra coming below, in this example we also assume that, upon default, bondholders get zero.[8]

## 2.2 Measuring Risk Prices

It is possible to model the market value of the debt in the illiquid regime as being determined by a discount over the value of a similar $(b, \eta, K)$ debt with zero risk, that is, debt with no liquidity or strategic risk.

Formally, let the market value of the debt be denoted $L(x; b, \eta, K)$. Without loss of generality we can define two "prices" or "discount factors" which we view as capturing the way the market prices the illiquidity risk and the strategic risk. We denote by $Q$ our measure of the price of illiquidity risk, and we label $S$, our measure of the price of strategic risk.

Given a $(b, \eta, K)$ bond the market price of such a bond in the absence of illiquidity and strategic risk, that is if $Q = S = 0$, is

$$B^* = \frac{b + \eta K}{r + \eta}.$$

Given this definition we can view the market value of the debt as a discount relative to the riskless version. Thus, our risk prices $Q$ and $S$ are such that

$$L(x; b, \eta, K) = B^* \left(1 - Q(b, \eta, K) - S(x; b, \eta, K)\right),$$

where, anticipating the results that follow, we specify that the price of illiquidity risk —the risk that the bond matures in the illiquid regime— is independent of potential earnings $x$.

There is a tight connection between these risk prices and the excess yield of the debt

[8] Assuming specific rules for the default payoff of the bonds simply adds a number of constants and it makes the algebra messier.

issued by the firm. To see this, let's denote the yield of debt with market value $L(x;b,\eta,K)$ as the value of the discount rate, $\tilde{r}$, that would make the value of a riskless version of the debt using that discount rate equal its market value. Thus,

$$\frac{b+\eta K}{\tilde{r}+\eta} = L(x;b,\eta,K) = B^* \left(1 - Q(b,\eta,K) - S(x;b,\eta,K)\right).$$

It follows—omitting the arguments in the functions—that

$$\tilde{r} - r = \frac{(r+\eta)\left(Q+S(x)\right)}{\left(1-(Q+S(x))\right)}.$$

The excess yield of the debt issued by the firm depends on the sum of the two risk prices. Any factors that increase this sum will result in higher excess yield.

We define the value of a $(b,\eta,K)$ debt that has **no illiquidity risk** as the value of the same $(b,\eta,K)$, denoted $L_{NI}(x;b,\eta,K)$, except that, if it matures in the illiquid regime, debtholders receive $K$. This is unlike the payoff of the debt issued by the firm that in that event (maturity in the illiquid regime) pays zero to debtholders. Simple calculations show that

$$L_{NI}(x;b,\eta,K) = L(x;b,\eta,K) + \frac{\eta K}{r+\eta+\upsilon},$$

and we view **the market value of the illiquidity risk** as the difference between the value of the debt with no illiquidity risk and the debt that has illiquidity risk, that is

$$L_{NI}(x;b,\eta,K) - L(x;b,\eta,K) = \frac{\eta K}{r+\eta+\upsilon}.$$

It follows that the price of illiquidity, $Q$, is

$$Q(b,\eta,K) = \frac{\eta K}{b+\eta K}\frac{r+\eta}{r+\eta+\upsilon}. \tag{1}$$

Using the definition of the risk prices, the price of strategic risk is

$$S(x;b,\eta,K) = \frac{b}{b+\eta K} + \frac{\eta K}{b+\eta K}\frac{\upsilon}{r+\eta+\upsilon} - \frac{r+\eta}{b+\eta K}L(x;b,\eta,K). \tag{2}$$

## 2.3 Properties of Risk Prices

**Price of Illiquidity Risk** There are two elements that influence the price of illiquidity risk in equation (1). First, the degree of backloading of debt payments —as measured by the ratio $\eta K/(b+\eta K)$— has a positive impact on the market price of risk. Second, the maturity of the debt —as measured by $\eta$— and the maturity structure of the underlying assets —as measured by $\upsilon$— also influence the market price of illiquidity risk. It is straightforward to see that:

- **Expected Maturity of the Debt.** Increases in expected maturity of the debt (decreases in $\eta$) decrease the price of illiquidity risk. In particular, $\lim_{\eta\to\infty} Q = 1$ and $\lim_{\eta\to 0} Q = 0$. That is, short term debt bears a high implicit price of illiquidity risk while consols are priced as if there is no illiquidity risk.

- **Asset Maturity**. Increases in the expected duration of the illiquid regime $(1/\upsilon)$ have similar effects: when the illiquid regime is arbitrarily short $(\upsilon \to \infty)$, then $Q = 0$. At the other end when the illiquid regimes lasts forever $(\upsilon = 0)$ then the price of illiquidity risk is proportional to the degree of backloading.

Before discussing the price of strategic risk it is useful to be explicit about the valuation of the debt. As in section 2, let $x^*$ be the lowest value of income that triggers default during the liquid regime. This threshold depends on the structure of the debt —the values of $(b, \eta, K)$— as well as the parameters defining the economic environment, $(\sigma, \upsilon, \mu, r)$, but to keep the notation simple we do not make this dependence explicit.

**Price of Strategic Risk** To study the properties of $S(x; b, \eta, K)$ we first need to establish some properties of the market value of the debt. As shown in the previous section, the default rule at maturity in the liquid regime is to default whenever $x < \bar{x}(K) = (r-\mu)K$, while the optimal rule before maturity is to default the first time that $x$ drops below $x^*$. This last event is associated with the stopping time $T_{x^*}$ given by

$$T_{x^*} = \inf\{t : t \geq T\upsilon \text{ and } x_t \leq x^*\}$$

The value of the bond once the firm is in the liquid regime, denoted $B(x; b, \eta, K)$, is given by

$$B(x; b, \eta, K) = E\left\{ \int_0^{T_\eta \wedge T_{x^*}} e^{-rt} b dt + e^{-rT_\eta} [K \aleph_{[(T_\eta < T_{x^*}) \cap \left(x_{T_\eta} \geq \bar{x}(K)\right)]}] \mid x_0 = x \right\}.$$

It is immediate to show that $B(x)$ —again suppressing most of the arguments— is an increasing function of $x$ (see the explicit solutions in Appendix C). The value of the debt in the illiquid regime —which is the relevant one to study the choice of maturity since it is at this stage that the firm selects the structure of the debt— is given by

$$L(x; b, \eta, K) = E\left\{ \int_0^{T_\upsilon \wedge T_\eta} e^{-rt} b dt + e^{-rT_\upsilon} [B(x_{T_\upsilon}) \aleph_{[T_\upsilon > T_\eta]}] \mid x_0 = x \right\}.$$

It follows that the market value of the bond in the illiquid state also increasing in $x$, since this corresponds to a lower probability of default.

Equation (2) then implies that $S(x; b, \eta, K)$ is *decreasing* in potential earnings ($x$). Thus, our model implies that firms with better prospects (higher $x$) face lower prices of strategic risk. $S(x; b, \eta, K)$ also depends on the structure of the debt and the features of the economic environment.

We can now use the results in Appendix C to analyze the implications of changing some of the parameters. Since the details depend on the particular value of $x$, as well as the relationship between $x^*$ and $\bar{x}(K)$, here we report some general results. A full account of the details is in Appendix C.

We find that:

- **Uncertainty**. We find that when the degree of uncertainty —as measured by $\sigma$— is arbitrarily low then the market price of strategic risk depends on the face value of the bond.
  - If the face value, $K$, is low (the precise value is given in Appendix C) we show that,

$$\lim_{\sigma \to 0} S = \begin{cases} 0 & \text{for } x \geq x_I^* \\ \frac{\upsilon}{r+\eta+\upsilon}\left(1 - \left(\frac{x}{x_I^*}\right)\right) & \text{for } x \leq x_I^* \end{cases},$$

where $x_I^*$ is the level of income at which the firm decides to go bankrupt (see Appendix C).

If the face value, $K$, is high then we get that

$$\lim_{\sigma\to 0} S = \begin{cases} 0 & \text{for } x \geq \bar{x}(K) \\ \frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_M(x) & \text{for } \bar{x}(K) \geq x \geq x_{II}^* \\ \frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_L(x) & \text{for } x_{II}^* \geq x \end{cases},$$

where the nonnegative functions $\hat{S}_M(x)$ and $\hat{S}_L(x)$ are defined in Appendix C, and $x_{II}^*$ is the level of income that triggers bankruptcy in this case.

At the other end —when $\sigma \to \infty$— the price of strategic risk is higher and, independently of the state of the firm, satisfies

$$\lim_{\sigma\to\infty} S = \frac{\upsilon}{r+\eta+\upsilon}.$$

In general our numerical results show that $S$ is increasing in $\sigma$.

- **Asset Maturity**. Changes in the expected duration of the illiquid regime have the following impact on the price of strategic risk:

$$\lim_{\upsilon\to 0} S = \frac{\upsilon}{r+\eta+\upsilon} = 0$$

and for $x_j^* \in \{x_I^*, x_{II}^*\}$ (depending, as before, on the value of $K$)

$$\lim_{\upsilon\to\infty} S = \begin{cases} >0 & \text{for } x \geq \min(x_j^*, \bar{x}(K)) \\ 1 & \text{for } x \leq \min(x_j^*, \bar{x}(K)) \end{cases},$$

When the illiquid regime lasts a long time ($\upsilon \to 0$), the market value of the debt is governed by the illiquidity risk and, consequently the price of strategic risk is zero. At the other end, when the duration of the illiquid regime is arbitrarily short there is no illiquidity risk the price of strategic risk is high if $x$ is low as it implies immediate default. If $x$ is above the level that triggers instantaneous default then the price is lower but not zero since in this case the price of debt is given by the function $B(x)$.

- **Expected Maturity of Debt**. We find that

$$\lim_{\eta\to\infty} S = 0, \text{ and } \lim_{\eta\to 0} S > 0$$

  The market charges a positive price for the strategic risk in the case of long bonds ($\eta < \infty$). Our numerical results show that $S$ is decreasing in $\eta$.

How can understanding the factors that move the prices of the two risks help us understand the forces that influence debt maturity? Here we present a heuristic argument that suggests a possible connection between changes in the economic environment and the choice of maturity. The next section discusses similar forces in the using our calibrated model.

**Potential Earnings and Maturity** Consider a firm that has optimally chosen the maturity of its bond and suppose that there is an increase in the value of its potential earnings, $x$. How should the firm adjust the maturity of its debt if it could reissue it? A higher $x$ implies that the price of strategic risk, $S(x)$, is lower while the price of illiquidity risk, $Q$, has not changed. If the firm **decreases** the expected duration of the debt (higher $\eta$) this results in a lower price of illiquidity risk and, using the limiting results as good indicators of the direction of the change, a higher price of strategic risk. Thus, in a sense, lowering maturity allows the firm economize in the risk factor whose price has increased (strategic risk) at the cost of increasing the price of the other risk (illiquidity risk)

**Uncertainty and Maturity** Let us consider the impact of an increase in $\sigma$. According to our results (again using the limits as good indicators of movements over the whole range) this increases the price that the market changes for strategic risk, $S$, while the price of default risk, $Q$, is unchanged. What can the firm do to compensate for those changes? If the firm shortens the maturity of the debt (increases $\eta$) then it simultaneously it increases the price of illiquidity risk and decreases the price of strategic risk. Thus, as before, such a change in maturity has the effect of economizing on the factor that became more expensive (strategic risk) by increasing the amount paid for illiquidity risk. Changing the maturity of the debt is one way of accomplishing this.

In the following section we show the quantitative implications of a calibrated version of the model that allows for refinancing.

# 3 Model

There is a risk-neutral firm that has to incur an irreversible cost to implement an investment project. The returns of the project are completely described by a few parameters that determine the associated stochastic processes for output (or earnings). Net earnings are given by a stochastic process $x_t$ such that

$$dx_t = \mu x_t dt + \sigma x_t dW_t,$$

where $W_t$ is a Wiener process or Brownian motion, and $\mu$ ("the percentage drift") and $\sigma$ ("the percentage volatility") are constants. Thus, the the technology is described by $(\mu, \sigma)$.

The financial market relevant for this firm has two regimes. In the **liquid regime**, the firm has access to equity markets. The amount of time spent in this regime is random and distributed exponentially with parameter $\alpha$. Denote by $T_\alpha$ the time spent in the liquid regime. At time $T_\alpha$ the regime switches from liquid to illiquid.

We assume that if the firm defaults on its debt in the illiquid regime, bondholders receive $D_B$.[9] Similarly, if the firm defaults in the liquid regime, bondholders receive $D_E$.

If the debt matures in the illiquid regime it must be refinanced. If the firm cannot issue debt that raises enough revenue to pay off the original debt-holders, the firm will default. This is the *illiquidity risk* associated with the bond. In case of default, the bondholders obtain $D_B$ and the shareholders receive zero.

The cost of refinancing, a fixed cost, is denoted $C$. In the event that the debt has to be refinanced and there is no other debt outstanding, the firm chooses a new debt structure $(b', K', \eta')$ subject only to the restriction that the market value of the debt must be at least

[9]For example, this includes the case in which debt-holders receive a certain fraction of the value of the expected stream of income.

as high as the amount to be refinanced. Note that the firm can hedge the risk of not being able to refinance when it first issues debt by choosing the original $(b, K, \eta)$ bond such that its face value, $K$, is small.

In the liquid regime, depending on the values of $b$, the firm has incentives to strategically default either before the debt matures (when $x$ is low) or at maturity. Formally, this corresponds to the case in which a firm in the high growth regime the firm can issue equity to pay its creditors. We refer to this case as strategic default risk, or just *default risk*.

As is standard in these models the existence of debt financing could potentially lead to the liquidation of the firm and this is inefficient given our assumptions about the stochastic process for earnings. Thus, debt financing can potentially lead to inefficient liquidation, as in Diamond (1991) and Leland (1994).

## 3.1 Valuations in the Liquid Regime

The market value of a firm in the liquid regime that has issued a $(b, \eta, K)$ bond, is denoted by $T(x; b, \eta, K)$. To simplify the notation, we suppress the dependence on $(b, \eta, K)$.

Standard arguments show that $T(x)$ satisfies the following Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned} rT(x) &= x - b + \frac{\partial T}{\partial x}(x)\mu x + \frac{\partial^2 T}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ &+ \eta[\max(\frac{x}{r-\mu} - K, 0) - T(x)] \\ &+ \alpha[V(x) - T(x)], \end{aligned} \tag{3}$$

where $V$ is the market value of a firm in the illiquid regime, which will be explained later.

If $b = 0$, The boundary conditions are

$$\lim_{x \to 0} T(x) = 0, \text{ and } \lim_{x \to \infty} \frac{T(x)}{x} = 0,$$

while if $b > 0$, the optimal default before the debt matures is the first time that the process $\{x_t\}$ hits $x^*$, where $x^*$ satisfies the standard value matching and super-contact conditions given by

$$T(x^*) = 0, \text{ and } T'(x^*) = 0.$$

Let's denote the lowest value of $x$ that triggers bankruptcy **at maturity** by $\bar{x}$. Thus,

$$\bar{x}(K) = (r - \mu)K.$$

Of course, if the face value of the debt is sufficiently small, the above formula implies that $\bar{x}(K) < 0$ and there is no default at maturity. Appendix B discusses the details of the solution to equation (3). In general, it is possible —depending on the choice of debt structure— to observe the following cases:

- The debt is safe. This implies no default at any time.
- The debt is safe until it matures. This means that the firm will choose to honor the coupon but potentially default at maturity.
- The debt is not safe in any state. In this case—depending on $x$—the firm will choose to default before the debt matures—standard strategic default—or at maturity (a type of solvency default).

Now, to understand how firms choose maturity, it is important to consider the value of the bond. In the liquid regime, although firms do not issue bonds (equity is a better choice), this price is important because it affect the bond price in the illiquid regime, in which firms do issue bonds.

The value of a $(b, \eta, K)$ bond in the liquid regime (again suppressing the constants from the notation) satisfies

$$\begin{aligned} rB(x) = b + \frac{\partial B}{\partial x}(x)\mu x + \frac{\partial^2 B}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ + \eta[\Pi(x, K) - B(x)] \\ + \alpha[L(x) - B(x)], \end{aligned} \tag{4}$$

where $L$ is the value of the bond in the illiquid regime and

$$\Pi(x,K) = \begin{cases} K & if\ x \geq \bar{x}(K) \\ D_G & if\ x < \bar{x}(K) \end{cases}.$$

Moreover, if $b > 0$, then the choice of bankruptcy by the firm implies that

$$B(x^*) = D_G.$$

If $b = 0$, the value of the debt must satisfy $\lim_{x\to 0} B(x) = 0$. Since the value of risk free debt is given by

$$B^* = \frac{b + \eta K}{r + \eta},$$

then the no-bubble condition is simply

$$\lim_{x\to\infty} [B(x) - B^*] \leq 0.$$

## 3.2 Valuations in the Illiquid Regime

In order to describe the value of the firm and the debt in the illiquid regime, it is necessary to take a stand on the possibility that the firm has to refinance any debt that matures in the illiquid state. Let $M(x;K)$ be the value of a firm that has just refinanced a maturing bond with face value $K$.

Consider now the value of a firm that has issued a $(b,\eta,K)$ bond and has the function $M(x;K)$—which must be determined endogenously—as its continuation value after refinancing. We take that $M(x;K) = 0$ corresponds to default.

The valuation of the firm, $V(x;b,\eta,K;M(x;K))$, satisfies

$$\begin{aligned} rV(x) = z - b + \frac{\partial V}{\partial x}(x)\mu x + \frac{\partial^2 V}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ +\eta\left[\max\left(M(x,K),0\right) - V(x)\right] \\ +\upsilon\left[T(x) - V(x)\right]. \end{aligned} \tag{5}$$

In addition, the solution must satisfy the appropriate boundary conditions to rule out bubble solutions. An important property of the solution to this valuation problem is that the function $V$ is increasing in $M(x;K)$ for any given debt structure.

We now describe the market value of the debt, $L(x;b,\eta,K;M(x,K))$. First, note that the value of the bond in the event that it matures in the illiquid regime depends on the value to the firms of refinancing. Formally,

$$\Lambda(x,K) = \begin{cases} K & \text{if } M(x,K) \geq 0 \\ D_B & \text{otherwise.} \end{cases}$$

that is, the bond gets repaid if the value to the firm of refinancing is positive. If that is not the case, the firm defaults and the post-default value of the debt is denoted $D_B$.

Standard arguments (if the function $L$ is $C^2$) show that $L$ satisfies the following HJB equation[10]

$$\begin{aligned} rL(x) = b &+ \frac{\partial L}{\partial x}(x)\mu x + \frac{\partial^2 L}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ &+ \eta\left[\Lambda(x,K) - L(x)\right] \\ &+ \upsilon\left[B(x) - L(x)\right]. \end{aligned} \tag{6}$$

Since increases in the refinancing value of the firm, $M(x;K)$, increase the chance that a bond maturing in the illiquid regime will be repaid, it follows that the market value of the bond, $L(x;b,\eta,K;M(x,K))$, is also increasing in $M(x,K)$.

A firm that needs to refinance a bond with face value $K$ faces the following constraints. It can issue a new debt $(b',\eta',K')$ that raises $K$ plus the fixed cost $C$. If the market is unwilling to lend the necessary amount the firm must default. Thus, a first constraint is that the market value of the new debt $(b',\eta',K')$ must be greater than or equal to the amount to be refinanced plus the refinancing cost. This is simply

$$L(x;b',\eta',K';M(x,K')) \geq K + C \tag{7}$$

[10]The Feynman-Kac theorem can be used to show that, for fixed stopping times, $L$ is $C^2$ even though $\Lambda(x,K)$ is an indicator function.

Since in this sequential setting the possibility of repeated refinancing can give rise to a bubble-like solution, we also impose that a debt that has no chance of maturing during the illiquid regime and that it gets repaid with probability one, cannot exceed the value of the project if it was debt free. This value is

$$\hat{L}(b', \eta', K') = \frac{(r + \eta' + \alpha)}{(r + \upsilon)(r + \eta' + \alpha) - \alpha\upsilon} b' + \frac{\upsilon}{(r + \upsilon)(r + \eta' + \alpha) - \alpha\upsilon} (b' + \eta' K').$$

Formally, we impose that

$$\frac{x}{r - \mu} \geq \hat{L}(b', \eta', K'). \tag{8}$$

The left side of equation (8) is the value of a firm that has zero debt, while the right side is the value of a bond with no default risk but no option to be refinanced.

## 3.3 Choice of maturity

We can now describe the problem faced by a firm that has to choose a new debt structure to refinance maturing debt with face value $K$. The firm solves the following problem,

$$\sup_{(b', \eta', K') \in \Sigma(x, K, M)} V(x; b', \eta', K'; M(x; K')) \tag{9}$$

subject to

$$\Sigma(x, K, M) \equiv \{(b', \eta', K') : L(x; b', \eta', K') \geq K + C \text{ and (8) is satisfied}\}. \tag{10}$$

We assume that if $\Sigma(x, K, M)$ is empty, then the value of the firm is zero.

We find it useful to view the maximized value in equation (9) as defining an operator $H$. Thus,

$$H(M)(x, K) = \sup_{(b', \eta', K') \in \Sigma(x, K, M)} V(x; b', \eta', K'; M(x; K')). \tag{11}$$

The equilibrium refinancing function—which is also the value of the firm—is a fixed point of the operator $H$.

**Proposition 1** *There exists at least one fixed point of $H$.*

**Proof.** See Appendix G ■

The reason we cannot replace the sup operator with a max operator is that the function $M(x; K')$ is not necessarily continuous.[11]

The problem faced by a firm at the beginning of its life that has to finance its investment issuing debt is similar to the problem faced by the firm who refinances existing debt. If the cost of the project is $C_I$ and the firm has resources given by $\overline{O}$, then the problem of the firm is equivalent to that defined by equations (9) and (10) with $K = C_I - \overline{O}$.

# 4 Analysis

In order to better characterize the choice of maturity, we solve the model numerically and present a series of plots. The parameterization is shown in the Appendix.

## 4.1 Liquid regime

First, we focus on the liquid regime and show the shape of the functions of the value of the firm, $T$, and the bond, $B$, as a function of earnings given the debt characteristics $(b, \eta, K)$. There are two thresholds that define the interesting regions relative to default. The first threshold, $\bar{x}_L$, is such that for values of $x$ lower than $\bar{x}_L$ the firms prefer to default at maturity. This threshold has a simple expression because the firm compares the value of continuing with no debt and repayment, $x/(r-\mu) - K$, and the value of shutting down, which is zero. The second threshold, $\hat{x}_L$, is such that the firm prefers to stop paying the coupon, default, and shut down before the bond's maturity. Figure X shows two examples, one in which the bond is back-loaded and one in which it is front-loaded. If the bond is front-loaded, $\bar{x}_L < \hat{x}_L$ and the firms prefers to default before maturity. In contrast, if the bond payments are back-loaded, a firm with $x < \bar{x}_L$ continues making coupon payment but defaults at maturity.

During the illiquid regime the value of the firm, $V$, and the value of the bond, $L$, look similar than during the liquid phase. And there are also two default thresholds. However,

[11]We can show that $H$ maps lower semi-continuous functions into themselves with a discontinuity when there is default. In the continuous region the standard argument shows that the max is well defined.

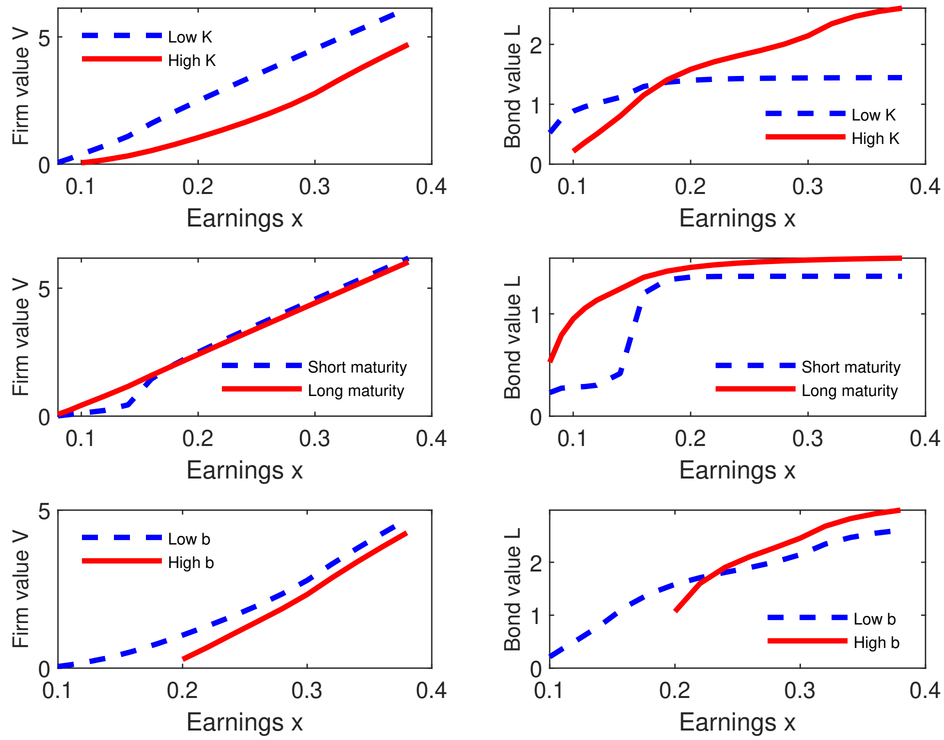



Figure 1: Value of the firm ($V$) and bond ($L$)

now, at the time of maturity, the comparison is between shutting down, with zero value, and rolling over the debt, which value depends on the function $M$. We also show that the thresholds, which are now $\bar{x}_I$ and $\hat{x}_I$, also change order depending if the debt is front-loaded or back-loaded.

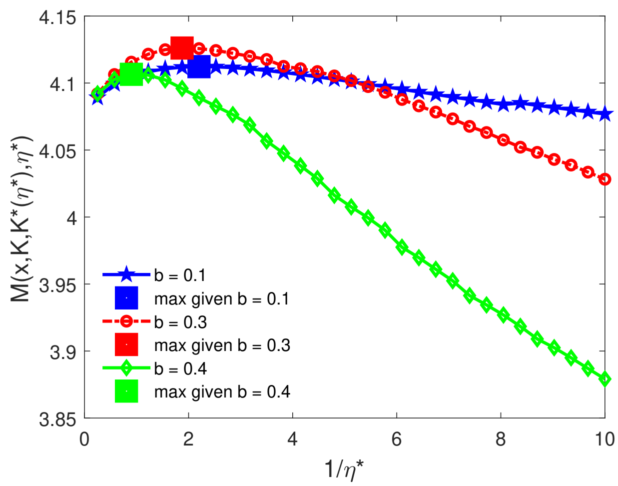



Figure 2: Choice of maturity and coupon

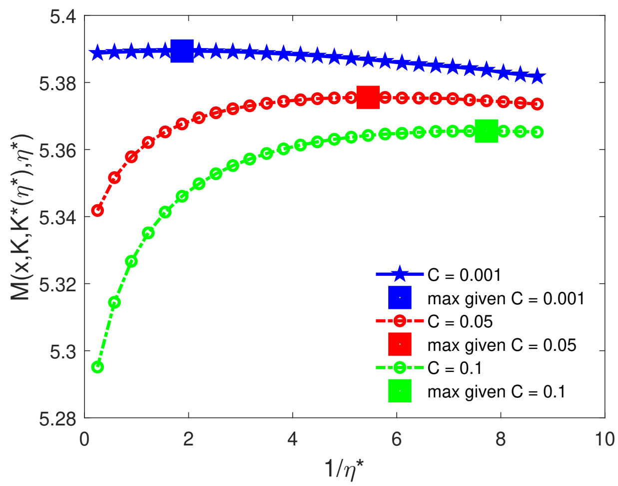



Figure 3: Choice of maturity and the issuing cost

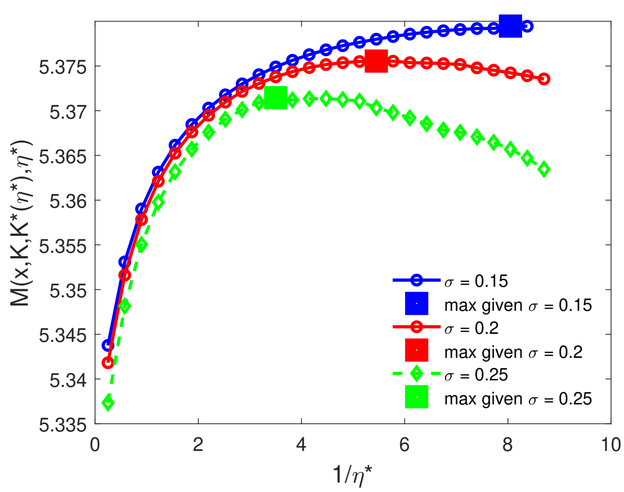



Figure 4: Cost of maturity and earnings risk

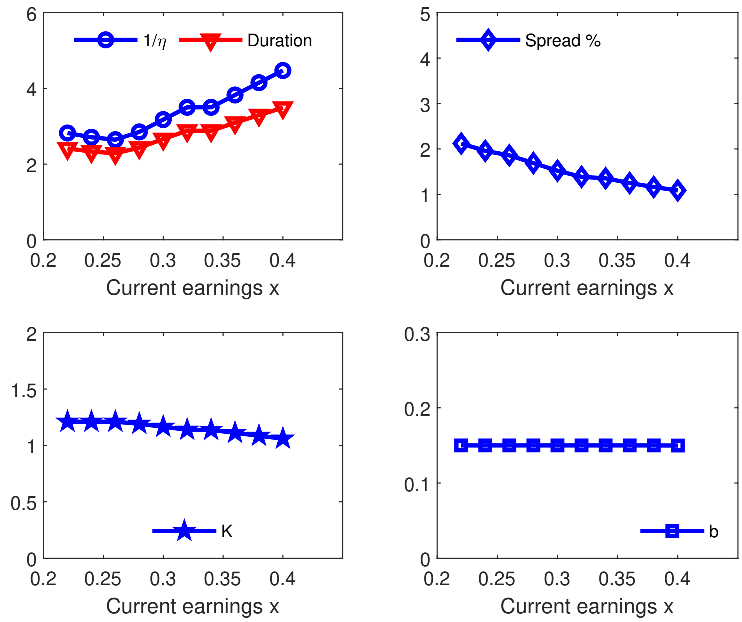



Figure 5: The role of duration of current earnings

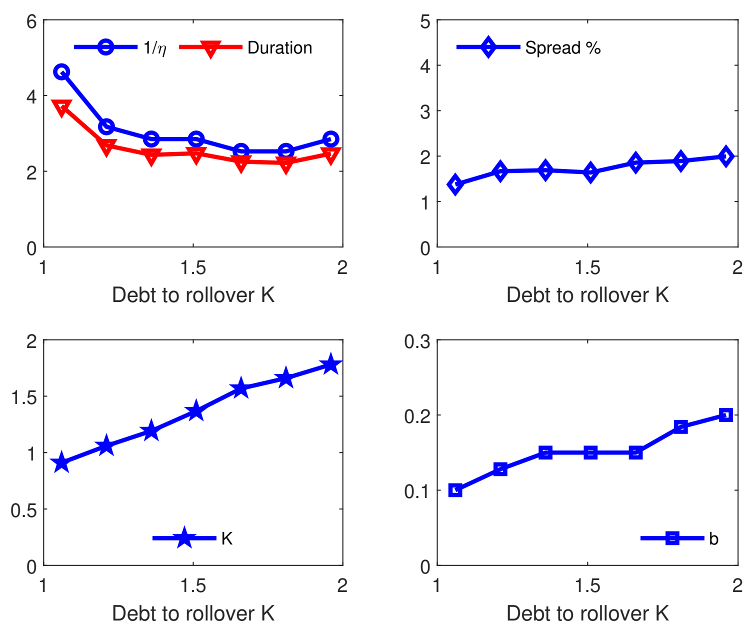



Figure 6: The role of the size of debt to rollover

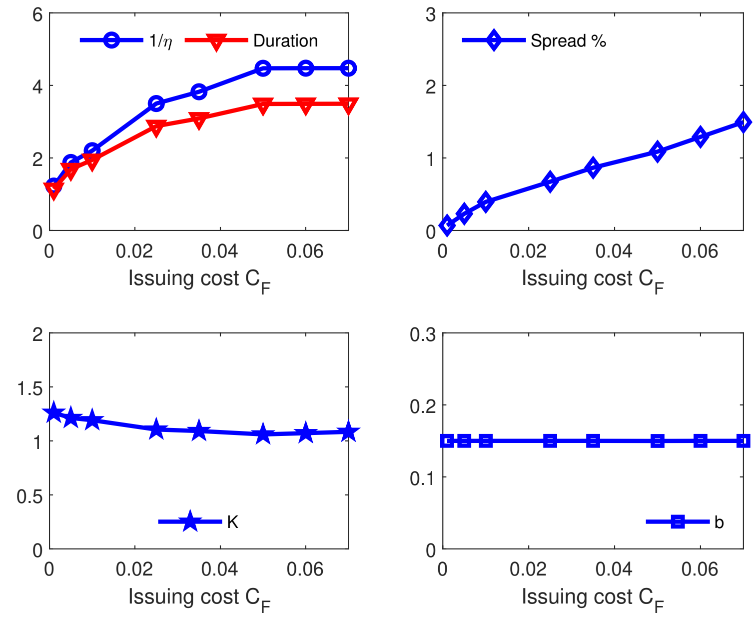



Figure 7: The role of issuing costs

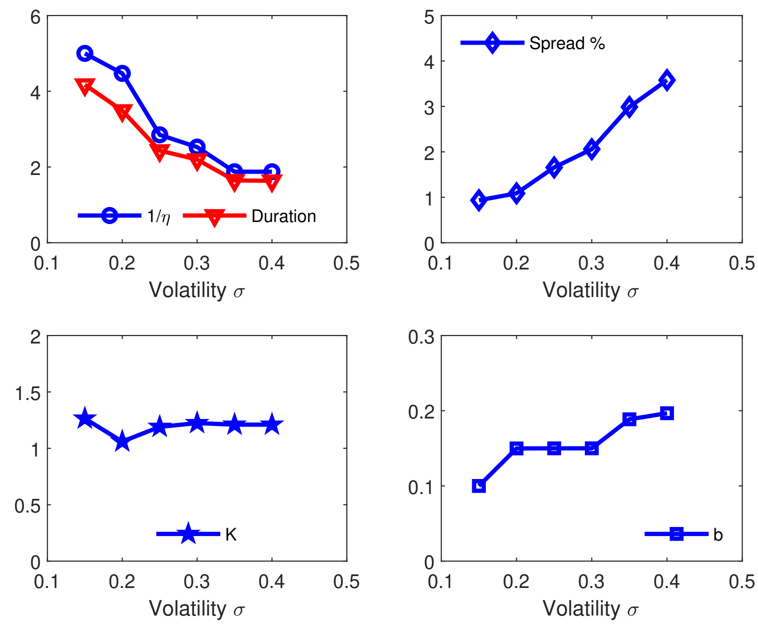



Figure 8: The role of earnings risk

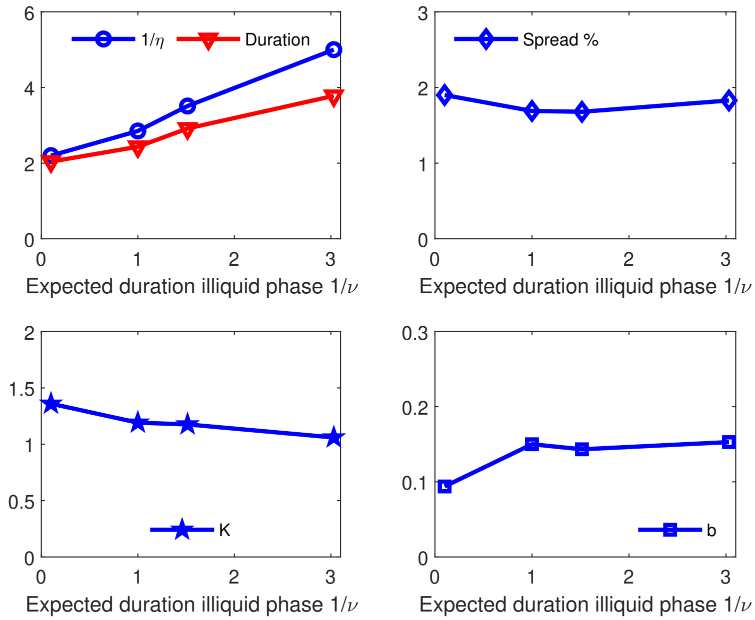



Figure 9: The role of duration of illiquidity

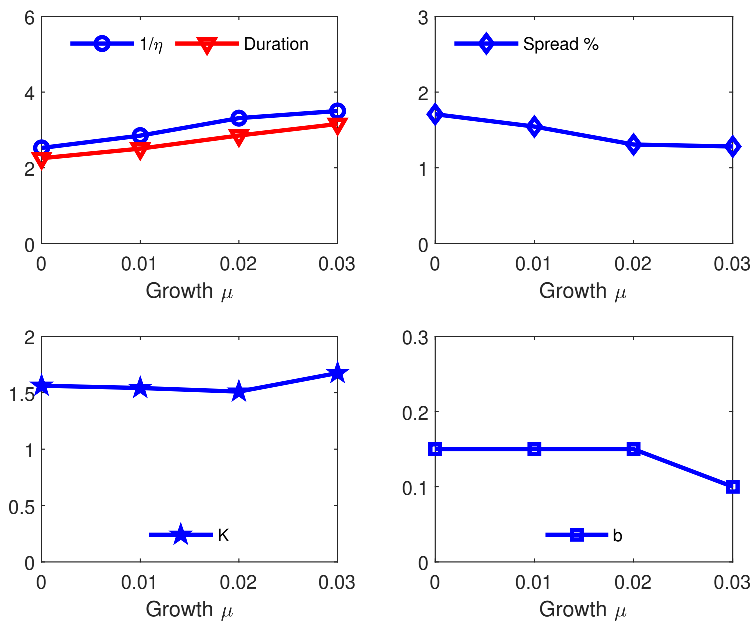



Figure 10: The role of duration of earnings growth

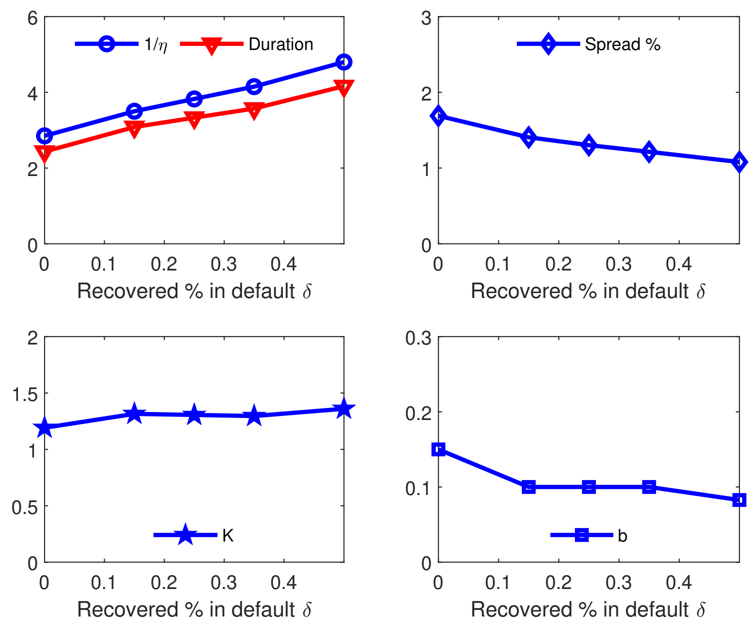


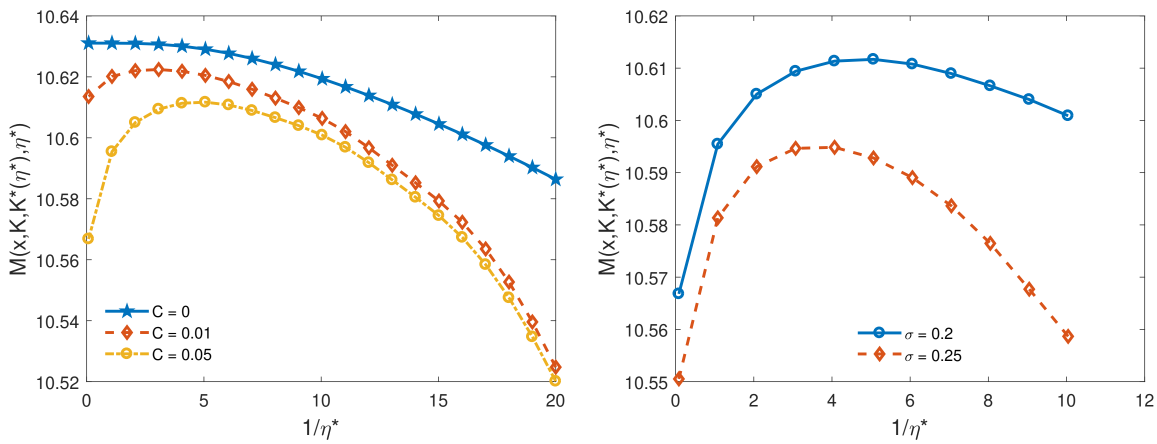

Figure 11: The role of duration of recovery in default

## 4.2 Choice of maturity

It seems useful to first discuss how arbitrary choices of maturity impact the value of the project and the set of feasible bonds. For doing this, we fix the value of potential earnings ($x$) and study how changes in maturity affect the value of a firm, denoted $M(x, K, K^*(\eta^*), \eta^*)$, that has to refinance a bond of a fixed value ($K$), for alternative values of the coupon.

Figure 1 shows the value of the firm, denoted $M(x, K, K^*(\eta^*), \eta^*)$, as a function of exogenously chosen maturity, $1/\eta^*$, for alternative values of the cost of refinancing, $C$, and measures of volatility of the firm's earnings, $\sigma$.

Figure 1: Maturity and the Value of the Firm




Cost of Refinancing ($C$) | Uncertainty ($\sigma$)

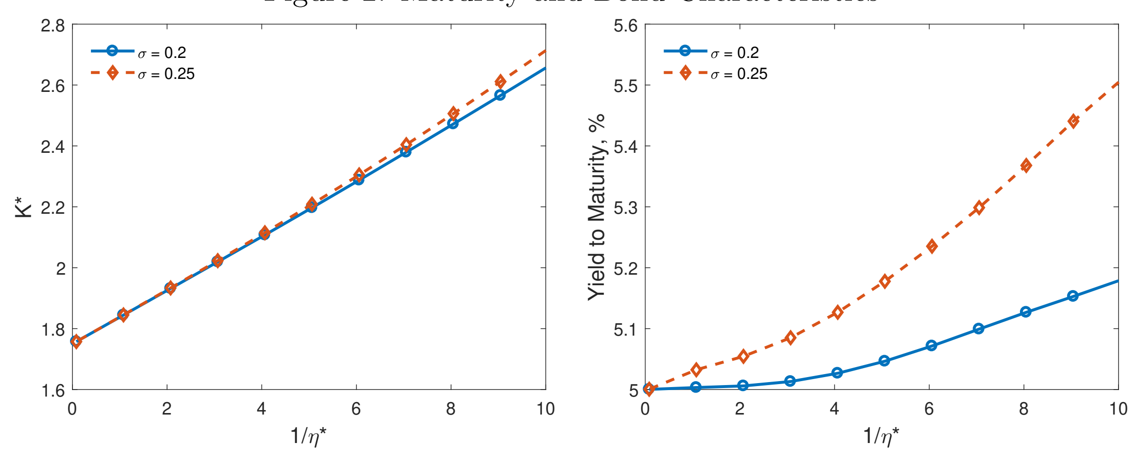
As expected, the function $M(x, K, K^*(\eta^*), \eta^*)$ is concave in maturity and, in all cases, the maximum is interior. Thus, the model is consistent with a well-defined optimal choice of maturity —the maximum of the $M$ function— even though all market participants are risk neutral and use the same discount factor. In the base case —$C = 0.05$ and $\sigma = 0.20$— the optimal debt structure has an expected maturity close to five years.

The left panel of Figure 1 summarizes the impact of changes in the cost of refinancing. If this cost decreases from the base case ($C = 0.05$) to $C = 0.01$ (a charge equal to 0.25% of the initial debt), the optimal maturity decreases from the base case of 5 years to less than 3 years. As the cost gets close to zero, the optimal maturity falls below 1 year. The lower costs of refinancing decrease the shadow cost of a bond maturing in the illiquid phase and, hence, decrease optimal maturity. We find that, under our parameterization, optimal maturity responds significantly to changes in the costs of refinancing.

The market value of a firm with a given stock of debt decreases as earnings volatility increases. The right panel of Figure 1 illustrates the impact of increasing volatility by 25% for a range of maturities from 1 month to 10 years. As in the case of the cost of refinancing, the effects are significant. The peak of the $M(x, K, K^*(\eta^*), \eta^*)$ function in the high volatility case is attained at a maturity slightly above 3.5 years, which corresponds to approximately a 25% decrease in optimal maturity when compared to the low volatility case.

How do the characteristics of the bond change with maturity? Figure 2 reports the face value —holding the market value of the bond constant— and the yield to maturity as a function of the expected maturity of the bond. If the firm chooses to issue longer bonds, then the face value must increase and the yield to maturity must also increase. The market prices long maturity bonds as riskier investments.

Figure 2: Maturity and Bond Characteristics




Face Value　　Yield to Maturity

The left panel of Figure 2 makes clear that the value maximizing debt structure **does not** pick the lowest face value, which would minimize the probability of strategic default since this is attained by the shortest possible bond. Similarly, the right panel of Figure 2 shows that the optimal bond **does not** minimize the interest cost as measured by the yield to maturity. For example, in the base case, the firm chooses to pay a small spread over the risk-free rate to issue debt with a longer maturity even though it had available the option of issuing shorter term debt at a rate close to zero.

The negative impact of earnings volatility also shows up in the bond characteristics. A firm in the higher variance environment ($\sigma = 0.25$) issues shorter debt that nevertheless pays a higher risk premium with a very small difference in the face value. The higher financing cost reflects that higher uncertainty implies a higher probability of strategic default, as well as lower likelihood that the firm will be able to refinance early maturing debt.

### 4.2.1 Maturity Choice and Value of the Firm

In this model, the value of a 100% equity-financed firm during the initial phase is

$$V^*(x) = \frac{z}{r+\nu}\frac{r+\nu\theta}{r} + \frac{\nu}{r+\nu-\mu}\frac{x}{r-\mu}.$$

In what follows, we interpret lower values of $V^*(x)$ as indicating (temporarily) poorly performing firms, while higher values characterize (temporarily) highly productive firms.

We find that in all experiments with pure discount bonds, optimal maturity increases with $V^*(x)$. We view this as evidence that firms that need to issue debt in "bad" times will **choose** to issue relatively short-term debt. In all the cases that we study, the firms could have issued longer term debt —say with the expected duration that it chooses when $V^*(x)$ is slightly higher— but the tradeoff between cost and risk makes this unprofitable.

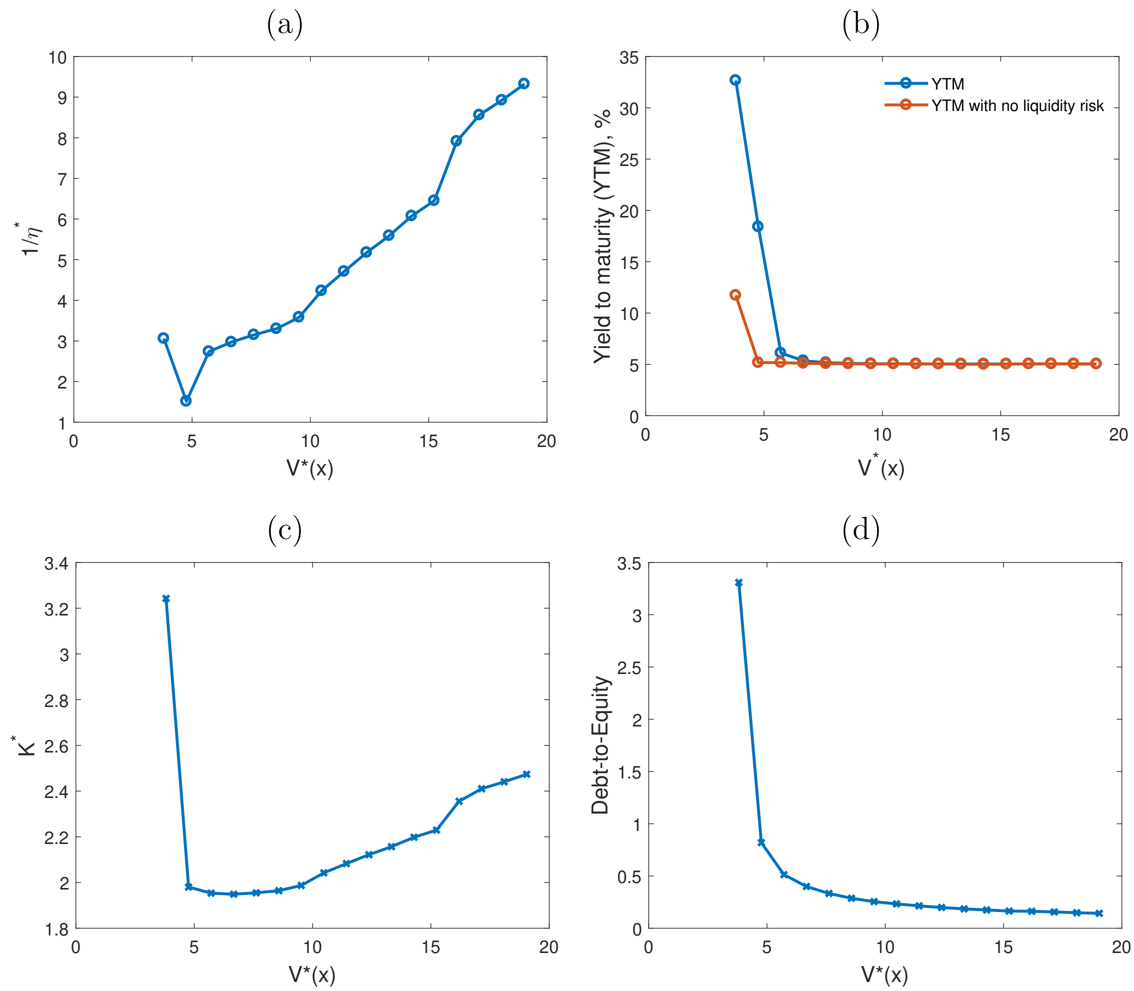
Figure 3 presents the values for key variables as potential output increases. First, Figure 3 (a) displays expected maturity. It is interesting to note what occurs at the lowest $V^*(x)$ for which financing $K$ is feasible (at that value of $x$, even an slightly larger value of $K$ cannot be financed). In this case, the cost of refinancing the debt relative to the potential earnings of the project becomes too high. The firm chooses to issue a very high face value—that could never refinanced—and 3 years of maturity. The yield to maturity, shown in Figure 3 (b), reaches more than 30%, indicating that this firm would be in financial distress. Note that we have also included in this figure the yield to maturity of the same bond but without liquidity risk.[12] This shows that as the risk of strategic default increases because $x$ decreases, the risk of default due to liquidity (which is captured by the difference between the blue and red lines) also increases. As we mentioned before, the choice of the bond is balancing the risk of

[12] This value is computed after solving for $L_{NI}$ numerically.

both types of default.

Beyond that very low value of $x$, expected maturity increases steadily from about 1 year to about 9 years as potential earnings increase. The face value, which is displayed in Figures 3 (c), shows that it follows a similar pattern as expected maturity, although less steep. As potential earnings increase and the firms decide that debt will be refinanced in the event that it matures during the illiquid phase, both yields to maturity decrease to a value close to 0%, as shown in Figure 3 (b). The debt-to-equity ratio is similarly patterned but with less curvature, where firms who would not refinance borrow as much as 3.3 times the amount of their equity, but the ratio decreases to less than half as potential earnings increases.

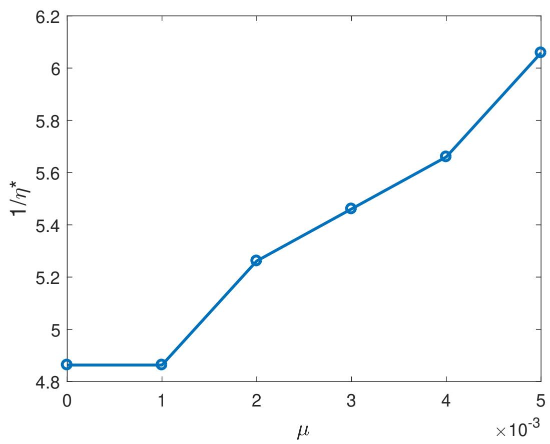
Figure 3: Optimal Debt as a Function of the State of the Firm



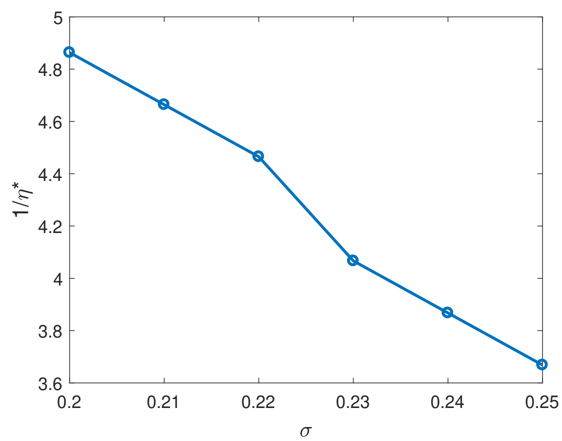
### 4.2.2 Technology and Equilibrium Maturity?

The earnings process of the firm is determined by the following four parameters: $\mu$, the expected growth rate; $\sigma$, a measure of the variability of the growth rate; $\upsilon$, an indicator of liquidity ($1/\upsilon$ is a measure of the expected duration of the illiquid phase); and $1-\delta$, which measures the fraction of the value of the firm that debt holders can appropriate in case of default. Figure 4 shows the impact of changes in the first two parameters on the equilibrium maturity of debt for the median firm in our calibration data.

Figure 4: Technology and Equilibrium Maturity I




Expected Growth Rate



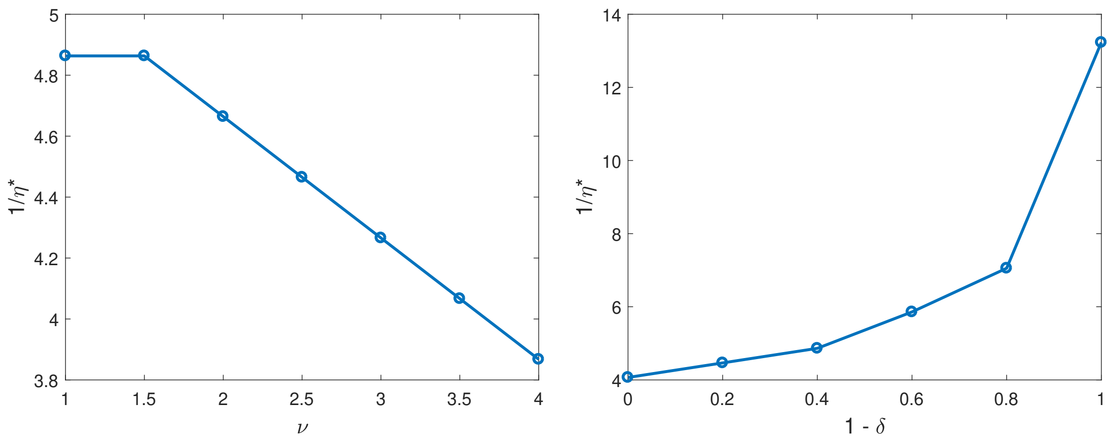

Variability Growth Rate

Firms with better growth prospects (higher $\mu$) issue longer debt. The intuition —as discussed in the case of a single bond— is that higher expected growth reduces the likelihood of default if the firm is in the liquid regime. Thus, for a fixed maturity, the market should lower the excess yield for longer maturities. The firm will find it optimal in this case to issue longer debt.

The impact of uncertainty is almost exactly the opposite. For a bond with a given face value, increases in $\sigma$ lower the expected value of the repayment in the liquid regime. This increases its price. The firm will then find it optimal to choose a shorter bond that can be refinanced at a time in which earnings are (potentially) higher.

Figure 5 summarizes how equilibrium maturity responds to changes in the other two technological parameters: $\upsilon$ and $\delta$.

Figure 5: Technology and Equilibrium Maturity II




(Inverse) Duration Illiquid Regime

Residual Value of Assets

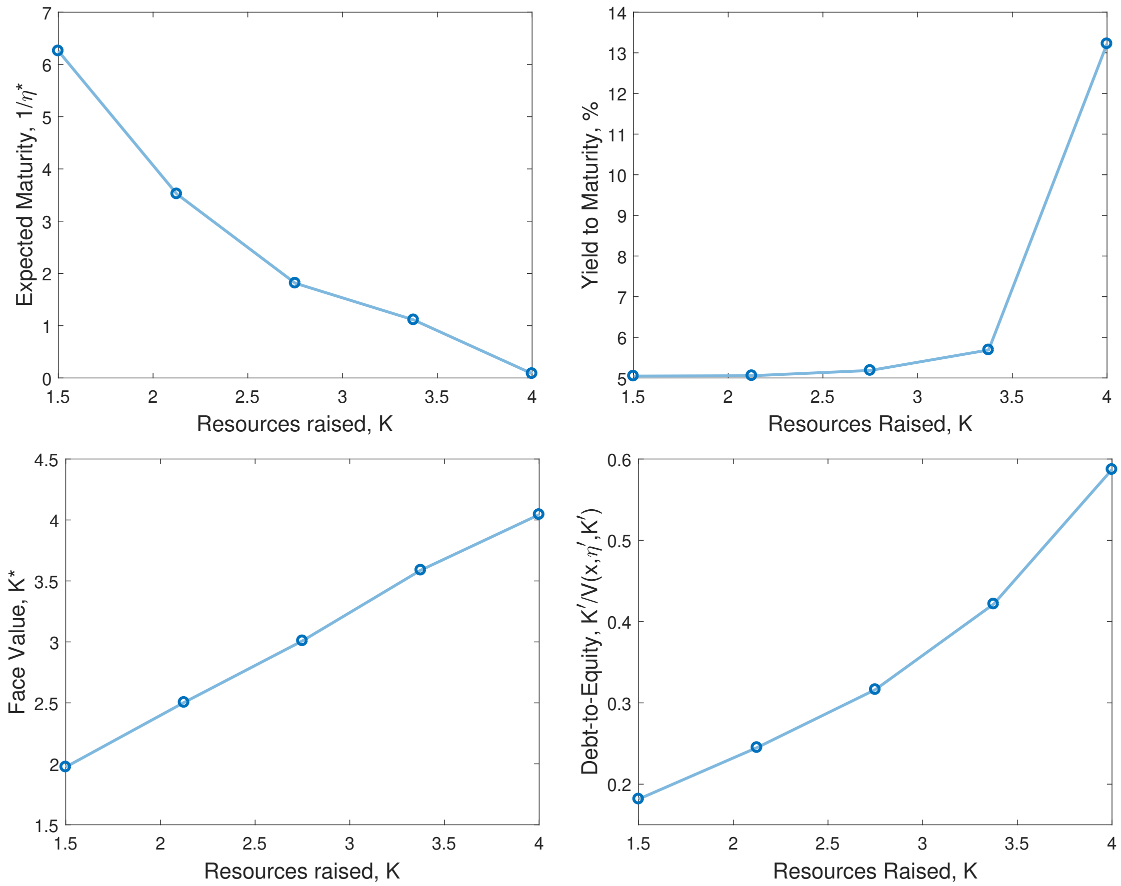
Consistent with the literature that emphasizes a connection between the maturity of the assets and the debt, our model implies that the shorter the duration of the illiquid regime, the shorter the expected duration of debt. The model also implies that the higher the resale value of the assets in case of default —that is, the higher $1-\delta$— the longer the optimal maturity. A higher $1-\delta$ is associated with lower costs (for the debt holders) of default in the liquid regime, and, hence, it is optimal for a firm to redesign its debt by lowering the implicit cost of illiquidity default. This is accomplished by increasing average maturity. Benmelech (2009) report a consistent result: American railroads that used rolling stock that could be more easily redeployed (higher resale value) tended to issue longer term bonds.

Appendix D shows how the other characteristics of the bond—its face value, yield spread, and debt-to-equity ratio—vary with these parameters. To the extent that differences in technology correspond to different sectors, the model implies that firms with similar values of expected earnings will issue debt with different characteristics.

### 4.2.3 Leverage

Figure 6 shows how the structure of debt varies with the financing needs of the firm. We interpret the funds raised by the firm, $K$, as an indicator of its leverage given that all other characteristics of the firm are constant. The model implies that firms with higher leverage issue shorter term bonds (panel (a)) with a higher face value (panel (c)). Increases in leverage appear to have a highly nonlinear impact on yields : As $K$ increases such that the debt-to-equity ratio increases from 15% to 30%, the model implies small changes in the spread. When $K$ increases beyond this threshold, increases in the debt-equity ratio from 30% to 40% are associated with large increases in the yield to maturity. Higher values of $K$ cannot be financed in the market.

Figure 6: Debt Structure and Leverage




### 4.2.4 Endogenous Borrowing Constraints

If firms would have access to perfect capital markets, all projects with $V^*(x) > K$ would be started with equity financing. In our setup, however, during the illiquid phase firms must raise $K$ with a defaultable bond. To illustrate to what extent this lack of access to equity markets would restrict start-ups, we computed the maximum value that a firm with a project valued $V^*(x)$ could raise. Figure 7 shows the maximum amount that could be raised relative to $V^*(x)$ and plot it as a function $V^*(x)$ (what could be raised in equity markets) for three alternative parameterizations.

Figure 7: Endogenous Borrowing Constraints

graphics/borrowing_const-eps-converted-to.pdf

There are at least two interesting findings. First, the maximum amount that could be raised is significantly lower than 1 and flat with respect to $V^*(x)$. This means that in our parameterization the financial friction of no access to equity market is very important for all size of projects. Notice that this ratio is an endogenous borrowing constraint that arises in our model as a consequence of lack of state-contingent debt instruments (firms can issue flexible bonds but not equity). Second, this constraint depends on the characteristics of the projects. Projects with higher salability or lower risk would have access to more credit, although their value $V^*(x)$ would not change.

## 4.3 Bonds with Coupons

The previous analysis was simplified because it assumed that there was no revenue during the first phase ($z = 0$) and, as a consequence, bonds were issued with no coupons, $b = 0$. In this section, we make minimal modifications to the values of the parameters to consider the role of bond coupons. In particular, the revenue during the first phase is increased to $z = 0.1$. Thus, firms can now issue a bond with coupons of up to $b = 0.1$.

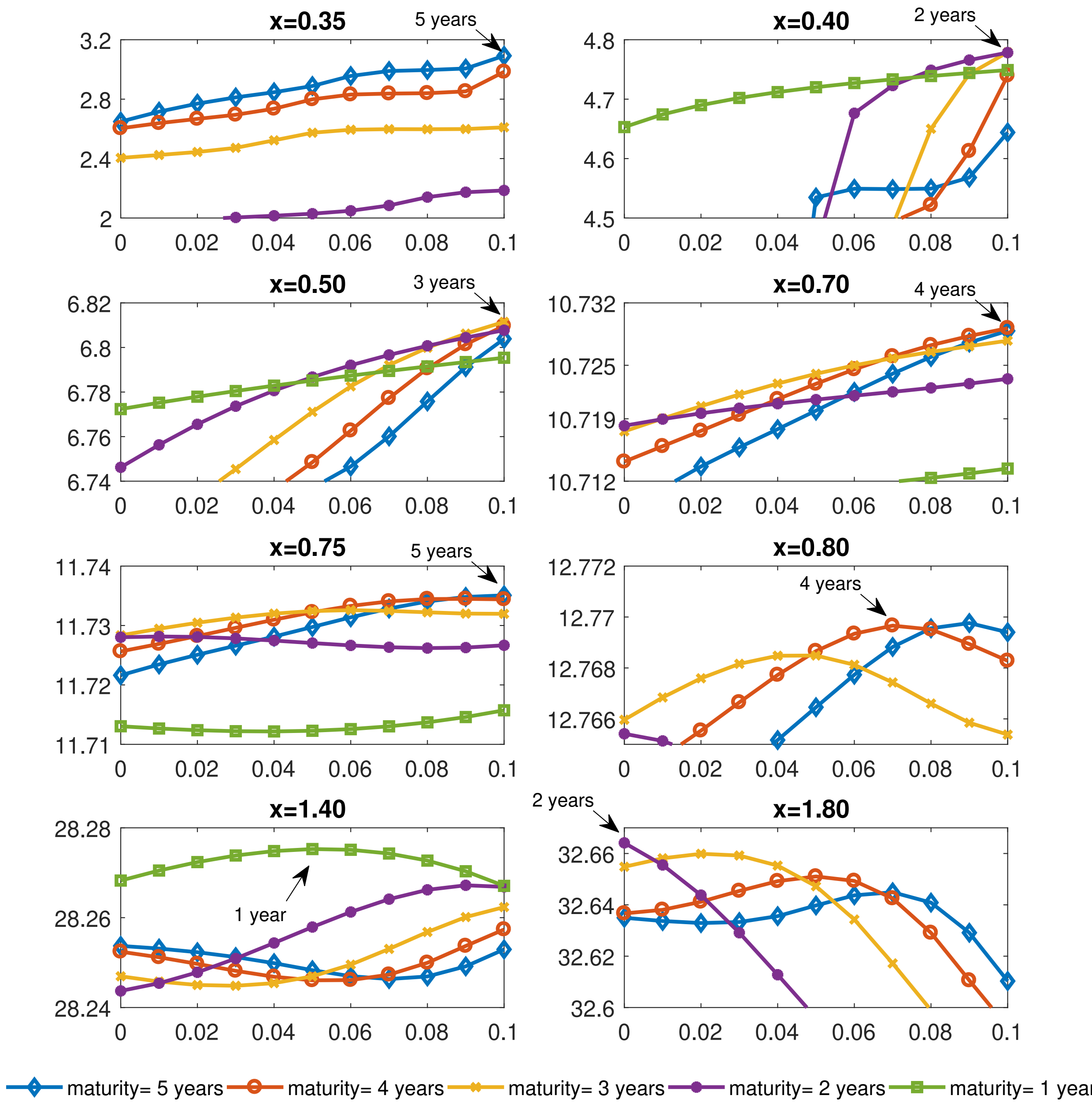
To show the role of the coupon, it is useful to show how arbitrary choices of the coupon impact the value of the project for a set of feasible bonds. Note now that there are three choices: $K$, $\eta$, and $b$. For each combination of $(\eta, b)$ considered, we select the value of $K$ such that the bond raises the desired resources. In particular, we analyze how changes in the bond coupon impact the value of a firm, denoted $M(x, K, K^*(\eta, b), \eta, b)$, that has to refinance a bond of a fixed value.[13]

The value of potential earnings ($x$) is set at many values from $x = 0.35$ and $x = 1.8$. Figure 8 shows the value of the firm for alternative values of maturity, $1/\eta$, exogenously chosen, as a function of the bond coupon. We find that for values of $x$ between 0.35 and 0.75 the largest possible coupon ($b = 0.1$) would be preferred. Actually, for values of $x$ between 0.35 to 0.70, reducing the coupon is worse for the value of the firm for any value of maturity (between 1 and 5) of the newly issued debt. For $x = 0.75$ we note that for some value of maturity, other values of the coupon are preferred. For instance, for a 2-year maturity bond, no coupon would be preferred. In terms of maturity, in this range of values of $x$ we see a pattern similar to Figure 3: Maturity initially decreases and then steadily increases.

Here, however, optimal maturity reaches 5 years at $x = 0.75$ and it starts to decrease. Given that in this example we are not allowing for bonds with maturity higher than 5 years, the alternative would be to keep maturity at 5 years and $b = 0.1$ and decrease $K'$. This is an option that we allow, but as Figure 8 shows, for values of $x > 0.75$, there are other better alternatives. For $x = 0.8$, maturity decreases to 4 years and the optimal coupon is 0.7. For $x = 1.4$, optimal maturity is 1 year and $b = 0.1$. As $b = 0$, maturity continues to increase with $x$ as in the case with no coupon. For instance, for $x = 1.8$ the optimal maturity is 2 years.

[13] We set $K = 2.8$.

Figure 8: Bond Coupon and the Value of the Firm




Even though we are not maximizing over the set of all possible bonds, our results suggest the following:

- For firms that have projects with low value, $V^*(x)$, it is optimal to issue bonds with a high coupon ($b = 0.1$ is the upper bound) and of short-to-moderate maturity. These firms face the highest price of strategic risk (driven by the low $x$), and they can economize on this factor by both frontloading the debt and choosing a relatively short maturity. For these firms, improvements in their prospects —as measured by $V^*(x)$— result in longer maturities. Thus, these results resemble the findings in the case of pure discount bonds.

- For firms with intermediate value, the results are more diverse. At the low end of this region, the best bond still frontloads payments because it chooses the highest possible coupon. However, since the higher $x$ implies a lower price of strategic risk, it is optimal to issue longer term debt (5 years). As the firm's prospects improve, the firms issue bonds with smaller coupons —which reduces the risk of strategic default— and they shorten the maturity.

- Finally, firms with very high values of $V^*(x)$ face low prices of strategic risk and issue zero coupon bonds with an intermediate maturity (2 years). This implies that the only risk bondholders face is the risk that, when the bond matures, the earnings will fall below $\bar{x}(K)$.

# 5 Conclusion

We study the optimal choice of the structure of debt by a risk-neutral agent —which we interpret as a firm— that borrows from a risk-neutral lender. The optimal maturity strikes a balance between two risks: the risk of default in the low growth (illiquid) regime and the risk of strategic default (in the high growth regime).

We find that the state of the firm —as measured by the unlevered value of its assets— and the characteristics of the technology (or the economic environment) influence the structure of the debt issued. In the case of pure discount bonds, as the level of (potential) output increases, the shadow price of the risk of strategic default goes down and the cost of default in the low growth regime goes up. Hence, it is optimal to lower the risk of default in the low growth (initial) regime by extending the maturity of the debt. The degree of uncertainty in the economic environment also influences the choice of financial structure. Higher uncertainty about growth rates is associated with shorter term debt. Investment projects with a higher value (to the lenders) upon default (e.g., higher resale value or lower cost of fire sales) and projects with lower leverage are financed with longer-term debt.

We also find that the results are sensitive to the assumption that debt can have any duration but that it cannot commit to paying intermediate coupons. When we analyze the more general case, we no longer find that the relationship between value and expected maturity is monotone. In our setting, if the firm is at a corner —meaning that it chooses to issue debt with the highest possible coupon— increases in value result in longer debt maturity. However, as the prospects of the firm improve, the optimal debt is less frontloaded and a higher share of the market value corresponds to the face value: Maturities are shorter. We interpret this as the consequence of smaller default risks associated with higher unlevered value of the firm. Finally, firms with very good prospects choose pure discount bonds.

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# Appendices

## A Maturity and Key Firms' Characteristics

In this appendix, we reproduce some previous empirical findings in the existent literature. We use the non-financial and non-regulated firms in Standard & Poor's Compustat from 1988 to 2006 to draw our sample of US firms.[14] The final sample size is 17,169 firm-year observations, with 3,023 different firms.

The construction of Compustat variables is the following[15]:

- Debt Ratio under 3 years*: A ratio of short-term debt (DD2 + DD3 + DLC) to total debt (DLTT + DLC).

- Debt Ratio under 5 years: A ratio of short-term debt (DD2 + DD3 + DD4 + DD5 + DLC) to total debt (DLTT + DLC).

- R&D Investment: A ratio of R&D expenditure (XRD) to sales (SALE). This variable captures "growth opportunities" following Guedes and Opler (1996).

- Volatility (Sales): A ratio of the standard deviation of the first difference in sales (SALE) to the mean of total assets (AT) across time weighted by the average total asset of the sample period, following Stohs and Mauer (1996) and Okzan (2002).

- Volatility (EBITDA): A ratio of the standard deviation of the first difference of EBITDA (SALE - COGS - XSGA) to the mean of total assets (AT) across time.

- Return Volatility: The standard deviation of stock return, where stock return is measured by the growth rate of adjusted closing price of stock (PRCCD/AJEXDI). This

[14] We have excluded financial firms from the data because the capital structure of the financial firms are influenced by different factors from other sectors. Similarly, we have excluded utilities because those firms are heavily regulated, and different debt maturity structure predictions may apply to such firms. We have also trimmed the extreme 1% of all independent variables to exclude outliers. We have discarded observations that have erroneous data for any of the variables used in the regression. To be specific, we have discarded any observation with short-term debt that is higher than total debt, and with current asset higher than total asset. See Stohs and Mauer (1996) and Johnson (2003) for similar practices.

[15] The (*) indicates variables that are included in the calibration target (Table 1A) but not in the regression (Table 2A).

is a measure of return volatility.

- Asset Maturity: The weighted average of short-term assets (ACT/COGS) and long-term assets (PPEGT/DP), where weights are defined, respectively, as (ACT/AT) and (PPEGT/AT), following Stohs and Mauer (1996).
- Asset Salability: A ratio of total net property, plant and equipment (PPENT) to total asset (AT), following De Jong et al. (2008).
- Kaplan-Zingales index: Following equation (12), where CF = (IB + DP), DIV = (DVC + DVP), C = (CHE), LEV = (DLTT + DLC)/(DLTT + DLC + SEQ), and Q = (LSE + CSHO*PRCC - CEQ -TXDB)/(LSE).[16] We interpret a higher value of this index as more reliance on external financing.
- Size: Logarithm of sales (SALE).
- Debt-to-Equity*: A ratio of total debt (DLTT + DLC) to equity (AT - DLC).
- Debt-to-Asset*: A ratio of total debt (DLTT + DLC) to assets (AT).
- Earnings Growth (Sales)*: A growth rate of sales (SALE)
- Std. Dev. of Earnings Growth (Sales)*: The standard deviation of the growth rate of sales (SALE).

We use the short-term debt ratio calculated as the ratio of debt that matures within a 5-year period to the total debt as the endogenous variable. Table 1A reports the descriptive statistics for the constructed variables. Importantly, short-term debt is prevalent among these firms: On average, 25 percent of the debt has maturity longer than 5 years. There is

[16] The index is defined as following:

$$KZ_{it} = -1.002\frac{CF_{it}}{AT_{it-1}} - 39.368\frac{Div_{it}}{AT_{it-1}} - 1.315\frac{C_{it}}{AT_{it-1}} + 3.139Lev_{it} + 0.283Q_{it}, \tag{12}$$

where CF is the cash flow, Div is the dividend, C is the cash and short-term investment, Lev is the leverage, Q is the Tobin's Q and AT is the lagged total asset, following the paper by Lamont et al. (2001). The paper argues that these variables obtained from the restricted version of the ordered logit of the central regression by Kaplan and Zingales (1997) provide a good explanation for the wedge that firms face between internal and external costs of funds.

also large variation in debt maturity, as shown by the fact that the standard deviation of the short-term debt ratio is 30 percent.

**Table 1A: Descriptive Statistics**

| | Mean | Median | Std. Dev. |
|---|---|---|---|
| Debt Ratio under 5 years | 0.750 | 0.892 | 0.300 |
| R&D Investment | 0.103 | 0.040 | 0.335 |
| Volatility (Sales) | 0.280 | 0.213 | 0.220 |
| Volatility (EBITDA) | 0.113 | 0.075 | 0.118 |
| Return Volatility | 4.023 | 3.894 | 1.604 |
| Asset Maturity | 6.490 | 5.398 | 4.650 |
| Asset Salability | 0.221 | 0.201 | 0.129 |
| Kaplan-Zingales Index | -2.948 | -0.787 | 8.678 |
| Size | 5.134 | 4.984 | 2.288 |

In Table 2A, we report panel-data regression on the short-term debt ratio. We also show the "expected" sign given findings in previous empirical literature. In terms of the regressions, we show several columns to argue that considering alternative volatility measures does not alter the result in any significant way. We also show that we can control by sector or firm fixed effects. For some of the intuitions that will be derived later, variations across sectors are important, so we also show the results controlling only by year fixed effects.

We find that firms/sectors with:

- Higher growth opportunity $\rightarrow$ longer debt maturity.
- Higher volatility $\rightarrow$ shorter debt maturity.
- Longer asset maturity $\rightarrow$ longer debt maturity (although not significant).
- Higher asset salability $\rightarrow$ longer debt maturity.
- More reliance on external financing $\rightarrow$ shorter debt maturity (although significant only with variation across industries).

**Table 2A: Panel Data Fixed Effects Regression on 5-Years Short-Term Debt Ratios**

| | Expected Sign | (1) Coef./SE | (2) Coef./SE | (3) Coef./SE | (4) Coef./SE | (5) Coef./SE | (6) Coef./SE | (7) Coef./SE |
|---|---|---|---|---|---|---|---|---|
| R&D Investment | −, + | -0.0123** | -0.0102* | -0.0165*** | -0.0139** | -0.0149** | -0.0125** | -0.0045 |
| | | (0.0060) | (0.0062) | (0.0061) | (0.0062) | (0.0061) | (0.0062) | (0.0097) |
| Volatility (Sales) | + | 0.0838*** | 0.0945*** | - | - | - | - | - |
| | | (0.0145) | (0.0152) | | | | | |
| Volatility (EBITDA) | + | - | - | 0.1191*** | 0.1249*** | - | - | - |
| | | | | (0.0249) | (0.0270) | | | |
| Return Volatility | + | - | - | - | - | 0.0085*** | 0.0097*** | - |
| | | | | | | (0.0031) | (0.0034) | |
| Asset Maturity | − | -0.0011 | -0.0009 | -0.0012 | -0.0010 | -0.0011 | -0.0009 | -0.0005 |
| | | (0.0008) | (0.0009) | (0.0008) | (0.0009) | (0.0008) | (0.0009) | (0.0013) |
| Asset Salability | − | -0.1997*** | -0.1596*** | -0.1989*** | -0.1601*** | -0.2019*** | -0.1631*** | -0.1042* |
| | | (0.0359) | (0.0403) | (0.0360) | (0.0403) | (0.0360) | (0.0404) | (0.0613) |
| Kaplan-Zingales Index | + | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0007* |
| | | (0.0003) | (0.0003) | (0.0003) | (0.0003) | (0.0003) | (0.0003) | (0.0004) |
| Size | − | -0.0424*** | -0.0404*** | -0.0420*** | -0.0405*** | -0.0411*** | -0.0394*** | -0.0346*** |
| | | (0.0019) | (0.0022) | (0.0020) | (0.0023) | (0.0025) | (0.0028) | (0.0082) |
| Year Fixed Effect | | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Industry Fixed Effect | | No | Yes | No | Yes | No | Yes | No |
| Firm Fixed Effect | | No | No | No | No | No | No | Yes |
| No. of Obs | | 17169 | 17169 | 17169 | 17169 | 17169 | 17169 | 17169 |
| $R^2$ | | 0.15 | 0.19 | 0.15 | 0.19 | 0.15 | 0.19 | 0.57 |

Note: Standard errors robust to heteroscedasticity are reported in parenthesis. ***, **, and * mean coefficients are significant at 1, 5, and 10 percent levels, respectively. Regression 1 and 2 use volatility of sales as a measure of risk, while regressions 3 and 4 use volatility of EBITDA as a measure of risk, and regressions 5 and 6 use return volatility as a measure of risk. Regression 7 does not contain any measure of risk because each firm corresponds to only one value of risk throughout the sample period.

These findings are quite similar to the articles discussed in section 2. However, our new theory offers an alternative explanation to the one usually provided in previous literature.

# B Additional Results: Valuations in the High Growth/Liquid Regime

**Preliminaries** Let $\lambda_i$ be a root of

$$r + \eta - \mu\lambda = \frac{\sigma^2}{2}\lambda(\lambda - 1).$$

Given our assumptions, it follows that $\lambda_1 < 0$ and $\lambda_2$. Moreover,

$$\lim_{\sigma\to\infty} \lambda_2 = 1 \text{ and } \lim_{\sigma\to\infty} \lambda_1 = 0,$$
$$\lim_{\sigma\to 0} \lambda_2 = \infty \text{ and } \lim_{\sigma\to 0} \lambda_1 = -\infty$$
$$\frac{\partial\lambda_2}{\partial\sigma} < 0 \text{ and } \frac{\partial\lambda_1}{\partial\sigma} > 0.$$

Similarly, let $\pi_i$ be a root of

$$r + \eta + \upsilon - \mu\pi = \frac{\sigma^2}{2}\pi(\pi - 1),$$

and, as before, denote $\pi_1$ as the negative root and $\pi_2$ as the positive root. Simple calculations show that the impact of changes in $\sigma$ upon the $\pi_i$ mimic their effect on the $\lambda_i$. It also follows that

$$\pi_1 < \lambda_1 \text{ and } \pi_2 > \lambda_2,$$

and that

$$\lim_{\upsilon\to\infty} \pi_2 = \infty \text{ and } \lim_{\upsilon\to\infty} \pi_1 = -\infty.$$

Recall that default at maturity happens if

$$x < \bar{x}(K, \theta z) = (r - \mu)\left(K - \frac{\theta z}{r}\right).$$

We assume that, upon default, bondholders get $(1 - \delta)$ of the value of the assets.

**Valuations** There are four possible cases depending on the parameters:

1. Case I: Perfectly safe bond: $b \leq \theta z$ and $K \leq \theta \frac{z}{r}$.

2. Case II: No default in the liquid regime except possibly at maturity: $b \leq \theta z$ and $K \leq \theta \frac{z}{r}$.

3. Case III: Default is possible in the liquid regime both before the debt matures and at maturity. However, it is never optimal for the firm to default at maturity: $b > \theta z$ and $K < \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta})$. This corresponds to the case $x^* > \bar{x}(K, \theta z)$.

4. Case IV: Default is possible in the liquid regime both before the debt matures and at maturity. This corresponds to $b > \theta z$ and $K \geq \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta})$ or, $x^* \leq \bar{x}(K, \theta z)$.

**Case I**. In this case, there is no risk and the values of the equity and debt are given by

$$B(x; I) = \frac{b + \eta K}{r + \eta} = B^*,$$
$$T(x; I) = \frac{x}{r - \mu} + \frac{\theta z}{r} - \frac{b + \eta K}{r + \eta}.$$

**Case II**. The value of the bond is given by

$$B(x; II) = \begin{cases} B^* + \bar{B}^1 II)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z), \\ \frac{b}{r+\eta} + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x}{r-\mu}\frac{r+\eta}{r+\eta-\mu}) + \bar{B}^2(II)(\frac{x}{\bar{x}})^{\lambda_2}, & x < \bar{x}(K, \theta z) \end{cases},$$

where

$$\bar{B}^1(II) = \frac{\eta}{r+\eta}\frac{1}{\lambda_1 - \lambda_2}\left[\frac{(1-\delta)\bar{x}}{r-\mu}\frac{(r+\eta)(1-\lambda_2)}{r+\eta-\mu} + \lambda_2(K - (1-\delta)\frac{\theta z}{r})\right],$$
$$\bar{B}^2(II) = \frac{\eta}{r+\eta}\frac{1}{\lambda_1 - \lambda_2}\left[\frac{(1-\delta)\bar{x}}{r-\mu}\frac{(r+\eta)(1-\lambda_1)}{r+\eta-\mu} + \lambda_1(K - (1-\delta)\frac{\theta z}{r})\right].$$

In this case the value of the firm is

$$T(x; II) = \begin{cases} \frac{x}{r-\mu} + \frac{\theta z - b}{r+\eta} + \frac{\eta}{r+\eta}(\frac{\theta z}{r} - K) + \bar{T}^1(II)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z), \\ \frac{x}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}^2(II)(\frac{x}{\bar{x}})^{\lambda_2} & x < \bar{x}(K, \theta z) \end{cases},$$

where

$$\bar{T}^1(II) = \frac{1}{\lambda_1 - \lambda_2}\left[(\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu})(1-\lambda_2) + \lambda_2\frac{\eta}{r+\eta}(\frac{\theta z}{r} - K)\right],$$
$$\bar{T}^2(II) = \frac{1}{\lambda_1 - \lambda_2}\left[(\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu})(1-\lambda_1) + \lambda_1\frac{\eta}{r+\eta}(\frac{\theta z}{r} - K)\right].$$

**Case III.** In this case we conjecture that the optimal default is such that $x^*$ —the level of $x$ that triggers default— is greater than $\bar{x}(K, \theta z)$. This implies that if, for some $t$, $x_t < x^*$, then the firm will default. However, if the stopping time $T_\eta$ precedes such an event, then the bond will be repaid in full. Thus, in this case, the strategic risk is concentrated on the period before the bond matures.

The relevant HJB equation for the value of equity solves the following:

$$rT(x; III) = x + \theta z - b + \eta[\frac{x}{r-\mu} + \frac{\theta z}{r} - K - T(x; III)]$$
$$+T'(x, III)\mu x + T''(x; III)\frac{\sigma^2}{2}x, \quad x \geq x^*,$$

with standard boundary conditions

$$T(x^*; III) = 0 \text{ and } \lim_{x\to\infty}\frac{T(x; III)}{x} < \infty$$

and the smooth pasting condition given by

$$T'(x^*; III) = 0.$$

The value of the firm is zero for $x < x^*$, and for $x \geq x^*$,

$$T(x; III) = \frac{x}{r-\mu} + \frac{\theta z}{r} - B^* + \underbrace{\left[\frac{1}{1-\lambda_1}(B^* - \frac{\theta z}{r})\right]}_{+}(\frac{x}{x^*})^{\lambda_1}.$$

The optimal bankruptcy decision rule is given by

$$x^* = \frac{(r-\mu)\lambda_1}{\lambda_1 - 1}(B^* - \frac{\theta z}{r}).$$

Since this case requires that $x^* \geq \bar{x}(K, \theta z)$, it implies that the face value of the bond must satisfy

$$K < \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta}) \equiv K_L$$

for the solution to be in this region. The value of the debt is

$$(1-\delta)\left[\frac{x}{r-\mu} + \frac{\theta z}{r}\right], \text{ for } x < x^*,$$

and

$$B(x; III) = B^* + \left[(1-\delta)(\frac{\theta z}{r} + \frac{x^*}{r-\mu}) - B^*\right](\frac{x}{x^*})^{\lambda_1}.$$

**Case IV**. If the face value of the bond is sufficiently high, then it is possible that the pre-maturity default threshold, $x^*$, is lower than the default threshold at maturity, $\bar{x}(K, \theta z)$. In this case, it is possible that default may occur before maturity (if $x$ drops below $x^*$ and the bond has not matured) or at maturity. The values of the firm and the bond are given by

$$B(x; IV) = \begin{cases} B^* + \bar{B}_H^1(IV)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z). \\ \frac{b}{r+\eta} + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x}{r-\mu}\frac{r+\eta}{r+\eta-\mu}) + \bar{B}_M^1(IV)(\frac{x}{\bar{x}})^{\lambda_1} + \bar{B}_M^2(IV)(\frac{x}{\bar{x}})^{\lambda_2}, & x^* < x < \bar{x}(K, \theta z) \\ (1-\delta)(\frac{x}{r-\mu} + \frac{\theta z}{r}), & x < x^*. \end{cases}$$

The constants solve the following system of equations,

$$\begin{aligned} \bar{B}_M^2(IV) &= \frac{1}{\lambda_1 - \lambda_2}[(1-\lambda_1)\frac{\eta(1-\delta)}{r+\eta-\mu}\frac{\bar{x}}{r-\mu} - \lambda_1 \frac{\eta}{r+\eta}((1-\delta)\frac{\theta z}{r} - K)], \\ \bar{B}_M^1(IV) &= (\frac{x^*}{\bar{x}})^{-\lambda_1}[-\bar{B}_M^2(IV)(\frac{x^*}{\bar{x}})^{\lambda_2} \\ &\quad + (1-\delta)(\frac{x^*}{r-\mu} + \frac{\theta z}{r}) - \frac{b}{r+\eta} - \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x^*}{r-\mu}\frac{r+\eta}{r+\eta-\mu})], \\ \bar{B}_H^1(IV) &= \bar{B}_M^1(IV) + \bar{B}_M^2(IV) + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + K\frac{r+\eta}{r+\eta-\mu}) - \frac{\eta K}{r+\eta}. \end{aligned}$$

The value of the firm is

$$T(x; IV) = \begin{cases} \frac{x}{r-\mu} + \frac{\theta z}{r} - B^* + \bar{T}_H^1(IV)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z) \\ \frac{x}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}_M^1(IV)(\frac{x}{\bar{x}})^{\lambda_1} + \bar{T}_M^2(IV)(\frac{x}{\bar{x}})^{\lambda_2}, & x^* < x < \bar{x}(K, \theta z) \\ 0, & x < x^* \end{cases},$$

with the constants being given by the solution of the following system (imposing boundary and smooth pasting conditions):

$$\bar{T}_M^2(IV) = \frac{1}{\lambda_1 - \lambda_2}[(1-\lambda_1)(\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu}) + \lambda_1 \frac{\eta}{r+\eta}(\frac{\theta z}{r} - K)],$$
$$\bar{T}_M^1(IV) = -(\frac{x^*}{\bar{x}})^{-\lambda_1}[\frac{x^*}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}_M^2(IV)(\frac{x^*}{\bar{x}})^{\lambda_2}],$$
$$\bar{T}_H^1(IV) = \frac{1}{\lambda_1}[\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu} + \lambda_1 \bar{T}_M^1(IV) + \lambda_2 \bar{T}_M^2(IV)].$$

The optimal default boundary solves

$$(x^*)^{\lambda_2}[\frac{\eta(r+\eta-\lambda_1\mu)}{(r-\mu)^{\lambda_2}(r+\eta-\mu)(r+\eta)}(K - \frac{\theta z}{r})^{1-\lambda_2}] = \frac{(1-\lambda_1)}{r+\eta-\mu}x^* + \lambda_1 \frac{b-\theta z}{r+\eta},$$

and it is immediate to verify that the solution is greater than $\bar{x}(K, \theta z)$ iff

$$K \geq \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta}) \equiv K_L.$$

# C The One Bond Case

In this section, we illustrate how the two prices of risk depend on properties of the economic environment. To simplify, we take a special version of cases III and IV. We assume that $\theta = 0$ —which corresponds to the case of no riskless income in the liquid regime— and $\delta = 1$, which implies that debt holders get nothing in the case of default. Allowing for a more general specification does not change the basic properties of the results but makes the algebra more cumbersome.

**Preliminaries** In this case the value of $K_L$ is given by

$$K_L = \frac{-\lambda_1 b}{(1-\lambda_1)\, r + \eta},$$

and default at maturity occurs if

$$x < \bar{x}(K) = K(r-\mu).$$

For $K \leq K_L$ —we label this case $I$—the bankruptcy threshold is

$$x_I^* = \frac{-\lambda_1}{1-\lambda_1} B^*(r-\mu),$$

while for $K \geq K_L$ —we label this case $II$— the bankruptcy threshold solves

$$\underbrace{\left[\lambda_1 \left(\frac{r-\mu}{r+\eta-\mu} - \frac{r}{r+\eta}\right) + \frac{\eta}{r+\eta-\mu}\right]}_{+} K(\frac{x_{II}^*}{\bar{x}})^{\lambda_2} = \underbrace{-\lambda_1 \frac{b}{r+\eta}}_{+} + \underbrace{\frac{(\lambda_1 - 1)}{r+\eta-\mu}}_{-} x_{II}^*.$$

**Properties of** $K$ These include

$$\lim_{\eta \to \infty} K_L = 0, \text{ and } \lim_{\eta \to 0} K_L = \frac{-\lambda_1(0)}{1-\lambda_1(0)} \frac{b}{r},$$

where $\lambda_i(0)$ is the value of the corresponding root when $\eta = 0$. Similarly, we use $\pi_i(0)$ as the value of the root when $\eta = 0$ as well.

Given the properties of $K_L$, there are some configurations that are not possible. For example, since $\lim_{\eta \to \infty} K_L = 0$, there is very short duration debt in case $I$, since for any finite $K$ it must be the case that —for a sufficiently short duration $1/\eta$— $K > K_L$. This case is also ruled out in environments with very high variance as $\lim_{\sigma \to \infty} K_L = 0$.

Similar considerations apply to $x^*$. For example, $x_I^*$ and $x_{II}^*$ both converge to zero as $\sigma \to \infty$ since the high option value of not defaulting dominates. Moreover, $x_{II}^*$ converges to zero when expected duration goes to zero. Finally, for consols (i.e., $\eta \to 0$) we have that $\lim_{\eta \to 0} x_I^* = \lim_{\eta \to 0} x_{II}^* > 0$.

**Risk Prices** The price of illiquidity risk, which we label $Q$, is independent of the state and given by

$$Q = \frac{\eta K}{b + \eta K} \frac{r+\eta}{r+\eta+\upsilon}. \tag{13}$$

The price of strategic risk during the illiquid regime —which is the relevant one when it comes to analyzing refinancing decisions— depends on whether the firm is in case $I$ or $II$ and on the particular value of potential earnings, $x$.

**Case** $I$ In this case, the prices of risk differ depending on whether $x$ is above —which we label $L^H$— or below —denoted $L^L$— the threshold that would trigger default in the liquid regime. Standard calculations show that

$$L_I^H(x) = B^* - \frac{\eta K}{r+\eta+\upsilon} - B^*\left(\frac{x}{x_I^*}\right)^{\lambda_1} + \bar{L}_I^H\left(\frac{x}{x_I^*}\right)^{\pi_1},\ x \geq x_I^*$$
$$L_I^L(x) = \frac{b}{r+\eta+\upsilon} + \bar{L}_I^L\left(\frac{x}{x_I^*}\right)^{\pi_2},\ x \leq x_I^*.$$

where

$$\bar{L}_I^H = \frac{B^*}{\pi_2-\pi_1}\left[-\lambda_1 + \pi_2\frac{r+\eta}{r+\eta+\upsilon}\right],$$
$$\bar{L}_I^L = \frac{B^*}{\pi_2-\pi_1}\left[-\lambda_1 + \pi_1\frac{r+\eta}{r+\eta+\upsilon}\right].$$

**Case** $II$ In this case, the optimal default threshold before the debt matures, $x_{II}^*$, lies below the default threshold at maturity, $\bar{x}(K)$. This defines three regions in terms of the value of potential earnings that require different treatment. We define the market value of the debt $L_{II}^H(x)$ when $x \geq \bar{x}(K)$. For intermediate values of $x$ (that is, $\bar{x}(K) \geq x \geq x_{II}^*$), we denote the value of the debt in the illiquid region by $L_{II}^M(x)$. Finally, for low values of $x$ ($x_{II}^* \geq x$), the value of the debt is $L_{II}^L(x)$. Standard pricing arguments can be used to show that the market value of the debt in each of these regions is given by

$$L_{II}^H(x) = B^* - \frac{\eta K}{r+\eta+\upsilon} + \bar{B}_1^H\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{L}_{II}^H\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1},$$
$$L_{II}^M(x) = \frac{b}{r+\eta} + \bar{B}_1^L\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{B}_2^L\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{L}_1^M\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1} + \bar{L}_2^M\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$
$$L_{II}^L(x) = \frac{b}{r+\eta+\upsilon} + \bar{L}_{II}^L\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$

where the constants satisfy

$$\begin{aligned}
\bar{B}_1^H &= \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2-\lambda_1} - \frac{b}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\lambda_1} - \frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta},\\
\bar{B}_1^L &= \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2-\lambda_1} - \frac{b}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\lambda_1},\\
\bar{B}_2^L &= \frac{-\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta},\\
\bar{L}_{II}^H &= \bar{L}_1^M + \frac{\pi_2}{\pi_2-\pi_1}\frac{\eta K}{r+\eta+\upsilon},\\
\bar{L}_2^M &= \frac{\pi_1}{\pi_2-\pi_1}\frac{\eta K}{r+\eta+\upsilon},\\
\bar{L}_1^M &= \left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\pi_1}\left[\frac{-\lambda_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta}+\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\right)+\frac{\pi_2}{\pi_2-\pi_1}\frac{b}{r+\eta+\upsilon}\right],\\
\bar{L}_{II}^L &= \left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\pi_2}\left[\frac{\pi_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta+\upsilon}+\frac{\eta K}{r+\eta+\upsilon}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\pi_2}\right)\right.\\
&\quad\left.+\frac{-\lambda_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta}+\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\right)\right].
\end{aligned}$$

Given the market value of the bond, $L(x)$, the price of strategic risk is (see equation (6))

$$S(x;b,\eta,K) = \frac{b}{b+\eta K} + \frac{\eta K}{b+\eta K}\frac{\upsilon}{r+\eta+\upsilon} - \frac{r+\eta}{b+\eta K}L(x;b,\eta,K).$$

**Maturity and Risk Prices** In this section, we report the limiting values of the two risk prices in the case of a short bond ($1/\eta \to 0$) and a consol ($1/\eta \to \infty$). We find that

Table C.1: Maturity and Risk Prices

| | | $1/\eta \to 0$ | $1/\eta \to \infty$ |
|---|---|---|---|
| $x \geq x^*$ | $Q$ | 1 | 0 |
| $x \geq x^*$ | $S$ | 0 | $\left(\frac{x}{x_I^*}\right)^{\lambda_1(0)} - \frac{\left(\frac{x}{x_I^*}\right)^{\pi_1(0)}}{\pi_2(0)-\pi_1(0)}\left[-\lambda_1(0)+\pi_2(0)\frac{r}{r+\upsilon}\right]$ |
| $x \leq x^*$ | $Q$ | 1 | 0 |
| $x \leq x^*$ | $S$ | 0 | $\frac{\upsilon}{r+\upsilon} - \frac{\left(\frac{x}{x_I^*}\right)^{\pi_2(0)}}{\pi_2(0)-\pi_1(0)}\left[-\lambda_1(0)+\pi_1(0)\frac{r}{r+\upsilon}\right]$ |

.

**Uncertainty and Risk Prices** Table C.2 reports the behavior of the risk prices as a function of $\sigma$. Since $Q$ —the price of illiquidity risk— is independent of the degree of uncertainty, we only report the values for the price of strategic risk.

Table C.2: Uncertainty and the Price of Strategic Risk

| | | $\sigma \to 0$ | $\sigma \to \infty$ |
|---|---|---|---|
| $x \geq x_I^*$ | $S$ | $0$ | $\frac{\upsilon}{r+\eta+\upsilon}$ |
| $x_I^* \geq x$ | $S$ | $\frac{\upsilon}{r+\eta+\upsilon}\left(1-\left(\frac{x}{x_I^*}\right)^{\pi_2^+}\right)$ | $\frac{\upsilon}{r+\eta+\upsilon}$ |
| $x \geq \bar{x}(K)$ | $S$ | $0$ | $\frac{\upsilon}{r+\eta+\upsilon}$ |
| $\bar{x}(K) \geq x \geq x_{II}^*$ | $S$ | $\frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_M(x)$ | $\frac{\upsilon}{r+\eta+\upsilon}$ |
| $x_{II}^* \geq x$ | $S$ | $\frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_L(x)$ | $\frac{\upsilon}{r+\eta+\upsilon}$ |

where

$$\hat{S}_M(x) = -\frac{b}{r+\eta+\upsilon}\frac{1}{B^*} + \frac{\eta K}{B^*}\left[\frac{1}{r+\eta}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2^+} - \frac{1}{r+\eta+\upsilon}\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2^+}\right],$$

$$\hat{S}_L(x) = \frac{1-(r+\eta)}{r+\eta+\upsilon} + \frac{1}{B^*}\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2^+} \times$$

$$\left[-\frac{\eta K}{r+\eta+\upsilon} + \frac{b}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\pi_2^+} - \frac{b}{r+\eta+\upsilon}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\pi_2^+} + \frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\frac{\upsilon}{\mu}}\right].$$

In these formulas, we use $\lambda_2^+$ and $\pi_2^+$ as the values of the roots when $\sigma = 0$. They are given by

$$\lambda_2^+ = \frac{r+\eta}{\mu} \text{ and } \pi_2^+ = \frac{r+\eta+\upsilon}{\mu}.$$

**Illiquidity and Risk Prices** In the model, $1/\upsilon$ is the expected duration of the illiquid regime. The price of illiquidity risk responds to changes in $\upsilon$ in a very intuitive way:

$$\lim_{\upsilon\to 0} Q = \frac{\eta K}{b+\eta K}, \quad \lim_{\upsilon\to\infty} Q = 0, \text{ and } \frac{\partial Q}{\partial \upsilon} < 0.$$

Table C.3 reports the limiting values of the price of strategic risk:

| Table C.3: Illiquidity and the Price of Strategic Risk | | | |
|---|---|---|---|
| | | $1/\upsilon \to \infty$ | $1/\upsilon \to 0$ |
| $x \geq x_I^*$ | $S$ | 0 | $\left(\frac{x}{x_I^*}\right)^{\lambda_1}$ |
| $x_I^* \geq x$ | $S$ | 0 | 1 |
| $x \geq \bar{x}(K)$ | $S$ | 0 | $\tilde{S}_M(x)$ |
| $\bar{x}(K) \geq x \geq x_{II}^*$ | $S$ | 0 | $\frac{\eta K}{b+\eta K} - \tilde{S}_L(x)$ |
| $x_{II}^* \geq x$ | $S$ | 0 | 1 |

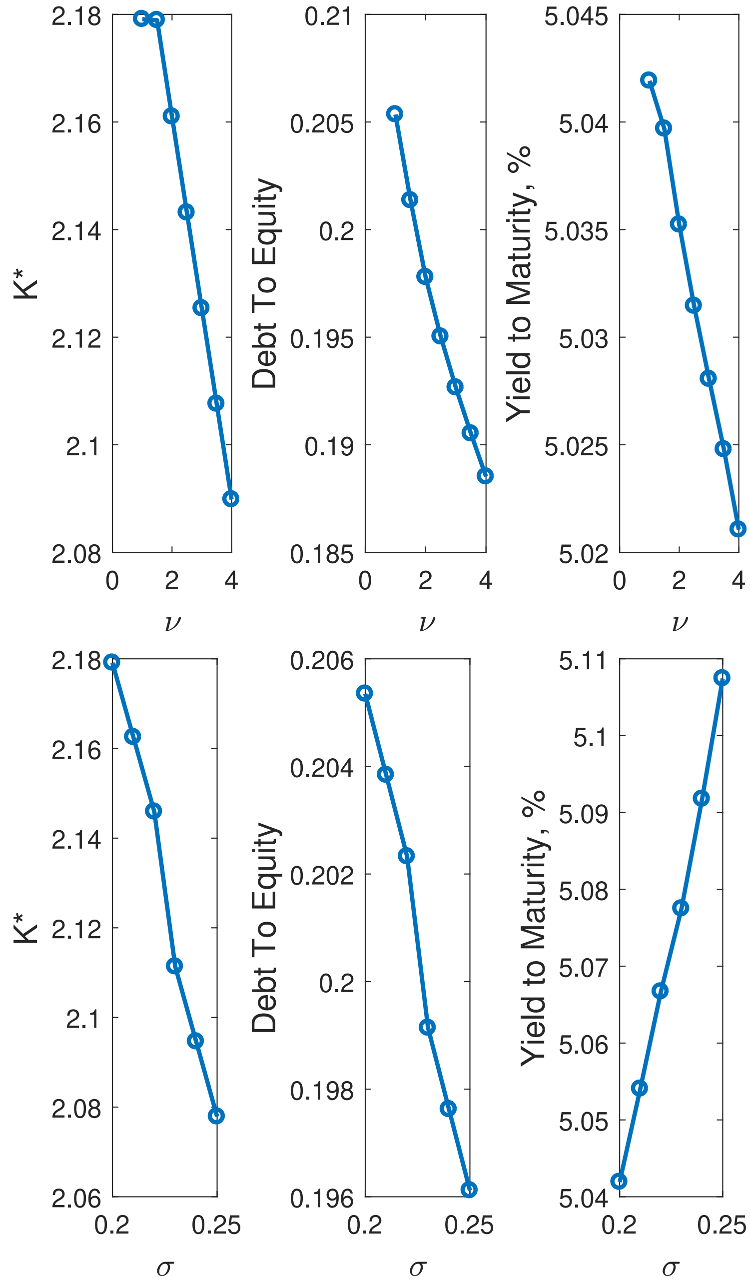
.

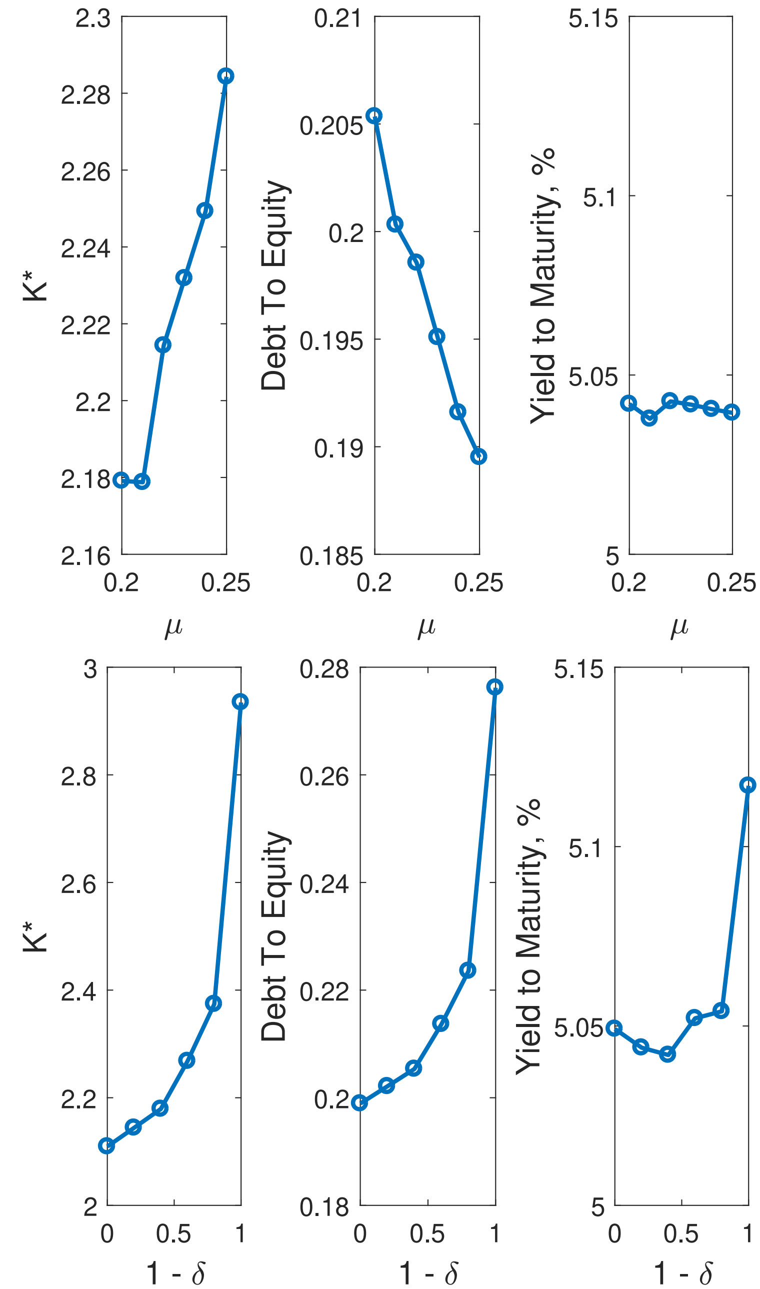
where

$$\tilde{S}_M(x) = \frac{-\lambda_1}{\lambda_2 - \lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} + \frac{b}{b+\eta K}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} + \frac{\lambda_2}{\lambda_2 - \lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1},$$

$$\tilde{S}_L(x) = \left[\frac{\lambda_1}{\lambda_2 - \lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} - \frac{b}{b+\eta K}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1}\right] - \frac{\lambda_1}{\lambda_2 - \lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2}.$$

# D Additional Graphs






# E Algorithm for Numerical Solution

The algorithm used to solve the model numerically is based on value function interation and it has the following steps:

1. Solve for the valuation of the firm $T(x, \eta, K)$ and bond $B(x, \eta, K)$ in the liquid regime using the solution given in Appendix C. Note that the valuation of the firm and bond

in the liquid regime does not change.

2. Solve for the valuation of the firm $V^1(x, \eta, K)$ and bond $L^1(x, \eta, K)$ in the illiquid regime using the solution given in Appendix C. This will served as an initial guess.

3. Use numericaal maximization to find the optimal value of the firm that needs to be refinanced such that $M^1(x; K) = \max_{(b', \eta', K')} V^1(x, b', \eta', K')$ such that $L^1(x; b', \eta', K') \geq K$.

4. Solve for $V^2(x, \eta, K)$ and $L^2(x, \eta, K)$ using $M^1(x; K)$, following the equations given in (**??**) and (5).

5. If $|V^2(x, \eta, K) - V^1(x, \eta, K)| + |L^2(x, \eta, K) - L^1(x, \eta, K)| < \epsilon$, finish. Otherwise, set $V^1(x, \eta, K) = V^2(x, \eta, K)$ and $L^1(x, \eta, K) = L^2(x, \eta, K)$, then back to step (3).

# F Optimal Maturity with Zero Cost

In this appendix we consider the case in which there is no cost in refinancing a loan and the safe income, $z$, is equal to zero. To simplify, we study only the case of a pure discount bond, that is, $b = 0$.

The conjecture is that in this case the optimal maturity will be zero (i.e., $\eta = \infty$). To check this conjecturem, we set the value of refinancing to the firm, $M(x)$, equal to what would be the value of a firm that issues zero maturity bonds. Thus, it keeps its principal constant and default only occurs in Phase II (high growth) whenever $x < \bar{x}(K)$. In particular, we conjecture that if the firm takes this continuation payoff as given and bonds are priced as if there is zero probability of default in Phase I (because of refinancing), then the optimal choice of a bond, the optimal $(\eta', K')$, equals $(\infty, K)$, where $K$ is the initial level of investment and, moreover, the resulting value of the firm under this choice is indeed equal to the conjectured $M(x)$.

First, we display the final equation (so that the problem can be computed), and then we describe the logic.

The value of the firm, $V(x;\eta,K)$, has two branches, one corresponding to $x \geq \bar{x}(K)$, which we label $H$, and the other corresponding to $x < \bar{x}(K)$, which we label $L$.

$$V^H(x) = \frac{x}{r-\mu} - \frac{\eta\upsilon}{r+\eta+\upsilon}\left(\frac{1}{r+\upsilon}+\frac{1}{r+\eta}\right)K$$
$$+ \bar{M}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\tau_1} + \bar{T}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{V}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1},$$

where

$$\bar{M}_1^H = \frac{1-\tau_2}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon-\mu} + \frac{\tau_2}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon}.$$

$$\bar{T}_1^H = \frac{1-\lambda_2}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta-\mu} + \frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta}.$$

$$\bar{V}_1^H = -\frac{1-\pi_2}{\pi_2-\pi_1}\frac{\eta+\upsilon}{r+\eta+\upsilon-\mu} - \frac{\pi_2}{\pi_2-\pi_1}\frac{r+\eta}{r+\eta+\upsilon}.$$

The lower branch is

$$V^L(x) = \frac{x}{r+\eta-\mu} + \frac{x\eta}{r+\eta+\upsilon-\mu}\frac{1}{r+\upsilon-\mu}$$
$$+ \bar{M}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\tau_2} + \bar{T}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{V}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$

where

$$\bar{M}_2^L = \frac{1-\tau_1}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon-\mu} + \frac{\tau_1}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon}.$$

$$\bar{T}_2^L = \frac{1-\lambda_1}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta-\mu} + \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta}.$$

$$\bar{V}_2^L = -\frac{1-\pi_1}{\pi_2-\pi_1}\frac{\eta+\upsilon}{r+\eta+\upsilon-\mu} - \frac{\pi_1}{\pi_2-\pi_1}\frac{r+\eta}{r+\eta+\upsilon}.$$

The value of the bond is given by the $L(x)$ function. As in the previous case, it has two branches:

$$L^H(x) = \frac{\eta}{r+\eta}K + \bar{B}_1^H\left(K-D_G\right)\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{L}_1^H\left(K-D_G\right)\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1},$$

where

$$\bar{B}_1^H = -\frac{\lambda_2}{\lambda_2 - \lambda_1}\frac{\eta}{r+\eta}.$$

$$\bar{L}_1^H = \frac{\pi_2}{\pi_2 - \pi_1}\frac{\eta}{r+\eta+\upsilon}.$$

The lower branch is

$$L^H(x) = \frac{\eta}{r+\eta}D_G + \frac{\eta}{r+\eta+\upsilon}(K - D_G) + \bar{B}_2^L (K - D_G)\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{L}_2^L (K - D_G)\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$

where

$$\bar{B}_2^L = -\frac{\lambda_1}{\lambda_2 - \lambda_1}\frac{\eta}{r+\eta}.$$

$$\bar{L}_2^L = \frac{\pi_1}{\pi_2 - \pi_1}\frac{\eta}{r+\eta+\upsilon}.$$

# G Proof of Proposition 1

Recall that the operator $H$ is given by

$$H(M)(x,K) = \sup_{(b',\eta',K')\in\Sigma(x,K;M)} V(x;b',\eta',K';M(x;K')),$$

where

$$\Sigma(x,K;M) \equiv \{(b',\eta',K') : L(x;b',\eta',K') \geq K + C_F$$
$$\text{and } \frac{x}{r-\mu} + \frac{r+\upsilon\theta}{r+\upsilon}\frac{z}{r} \geq \frac{b'}{r+\upsilon} + \frac{\upsilon}{r+\upsilon}\frac{b'+\eta K'}{r+\eta}\}.$$

Let $\geq$ be the natural order in the space of functions. Thus $F \geq G$ iff for all $(x,K)$ $F(x,K) \geq G(x,K)$. Since $M(x;K')$ is the continuation value of the firm, $V(x;b',\eta',K';M(x;K'))$ increasing in $M(x;K')$. Moreover, higher continuation values increase the likelihood that the debt will be repaid at maturity. Thus, the set $\Sigma(x,K,M)$ is also increasing (in the set inclusion sense) in $M(x;K')$.

Let $\mathcal{M}$ be the set of functions that satisfy

$$0 \leq M(x, K) \leq x/(r - \mu).$$

Then, it follows that $H$ maps $\mathcal{M}$ into itself. Moreover, $\mathcal{M}$ is a complete lattice and hence the Knaster-Tarski theorem implies that the set of fixed points is nonempty. This completes the argument.

# H Calibration

Table 1, is extremely simple since there are very few relevant parameters. We set the value of $\mu$ to zero as a normalization because the relevant variable for discounting is $r - \mu$.

**Table 1: Parameters**

| Parameters | Value | Basis |
|---|---|---|
| $(\mu, z, \theta)$ | 0 | Normalization |
| Risk free rate, $r$ | 0.05 | Standard |
| Refinancing cost, $C$ | 0.05 | Issuance cost-to-debt ratio |
| Volatility, $\sigma$ | 0.2 | Sales volatility |
| Arrival prod. phase, $\upsilon$ | 1 | Debt maturing under 5 years |
| Recovery, $\delta$ | 0.6 | Debt-to-equity |

There are 5 remaining parameters. The value of the cost of issuing debt, $C$, is set to 0.05, which is slightly above 1 percent of the amount of debt issue. The interest rate, $r$, is set to a standard value, 5 percent. The value determining the value of volatility of the process for $x$, $\sigma$, is calibrated a priori at 0.2 because that is the median value for the standard deviation of the first difference of sales (see Table 2). The value of the funds that must be raised, $K$, is chosen such that it coincides with the maximum that can be raised for the lowest $x$, $K = 1.68$. The remaining 2 parameters—the Poisson parameter determining the probability of arrival of the productive phase, $\upsilon$, and the share recovered of the value of the firm in case of default, $\delta$—are calibrated such that the model provides reasonable predictions for the debt-to-asset ratio and debt maturity. Note, however, that to compare the model with data we need to pick the value of $x$ that would represent the median firm. We pick the value of $x$ that best represents the state of the median firm in the data in terms of the debt-to-equity ratio.

**Table 2: Fit of Calibration Targets**

| | Model | Median Firm |
|---|---|---|
| Debt-to-Equity ratio | 0.205 | 0.208 |
| Debt-to-Asset ratio | 0.176 | 0.195 |
| Share of debt with maturity under 3 years | 0.465 | 0.606 |
| Share of debt with maturity under 5 years | 0.647 | 0.870 |
| Volatility (Sales) | 0.200 | 0.209 |

Note: See Appendix A for an explanation of the construction of these variables.