Central Bank's Response to a Negative Cash Flow or How to Inflate If You Must (preliminary and incomplete)*

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Abstract

Some central banks are required to transfer resources to the Treasury to finance deficits. From the perspective of cash flow, this requirement constitutes a negative cash flow in the central bank's balance sheet. Other central banks also experience a negative cash flow for a completely different reason: Increases in nominal interest rates in the presence of long dated securities and short dated liabilities result in a negative cash flow.

In this paper I derive the best strategy to finance a **temporary negative** cash flow of uncertain duration. I find that a robust feature of the optimal policy is that the amount of seigniorage raised —in the simple model from printing money— falls short of the financing needs. The difference is covered with borrowing at the market rate. This results in a pattern of increasing interest rates that stabilize (at a lower level) when the need to transfer resources ends.

If the penalty for large inflations (jumps in the price level) is finite, then countries with lower costs of high inflations will choose lower interest rate policies, which is a form of unpleasant monetarist arithmetic.

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1 Introduction

In January 2016 the newly elected government of Argentina announced a policy of decreasing government budget deficits in the context of a stabilization plan. It also specified what fraction of the deficit would have to be financed by the monetary authority. In the case of Argentina, this was a significant amount. For the year 2016 the Argentinean central bank (BCRA) —a non-independent central bank— was required to transfer to the Treasury approximately 2% of GDP out of a total government deficit of 4.8% of GDP. The plan specified that the transfers would decrease and, eventually, end.[1] This policy effectively required the BCRA to finance a negative cash flow in the form of a transfer to the Treasury.

In a different context —both geographically and institutionally— some independent central banks are faced with a similar problem: How to finance a negative cash flow. In the case of several independent central banks the negative cash flow is the consequence of the difference between the earnings on their portfolios of long dated securities and the interest payments on their short run liabilities (mostly reserves).[2] The Federal Reserve Bank (FED) has recorded negative cash flows since it started raising the policy rate and, at this time (April 2023), the negative balance is about \$44 billion. The Swiss National Bank (SNB) lost the equivalent of \$143 billion in 2022 which is about 18% of GDP, and the Reserve Bank of Australia has reported a negative equity of %12.4 billion due to the increase in nominal interest rates.

Although the origin of the negative cash flow in the two types of central banks is quite different —inflationary finance in one case and losses from interest rate risk in the other— they both face the same question: How should this temporary negative cash flow be financed? When a central bank has no fiscal support, that is, when it cannot expect to receive funds from the Treasury, the options are quite limited. Since the only genuine source of revenue that a central bank has is seigniorage, the question can be stated as when and how much seigniorage should be raised.[3]

Even before the recent increase in nominal interest rates the academic literature

[1] In addition, the BCRA decided to increase the monetary base by close to another 2.5% to increase the stock of foreign reserves. More recently (March 2023) the BCRA has announced a policy of decreasing transfers (Adelantos Transitorios) over the next two years.

[2] Since the 2008 crises central banks have held large portfolios of relatively long dated securities whose value have decreased as a consequence of increases in nominal interest rates. Their liabilities are short run and not affected by changes in interest rates

[3] In some cases the central bank can reasonably expect to generate a surplus from the earnings of its portfolio in “normal” times. In addition, if there is growth then the time path of seignorage is increasing over time.

noted that independent central banks faced a type of maturity transformation risk. The literature concentrated on studying the feasibility of a given dividend and monetary policy rule. Put differently, research by Del Negro and Sims (2015) and Hall and Reis (2015) took the monetary policy rule as given —typically a version of the Taylor rule— and evaluated the probability that a central bank could maintain the given policy in the face of changes in the net income from their portfolio. Even though the results depend significantly on assumptions about seigniorage, the findings were mostly reassuring in the sense that the probability of not being able to implement a given policy rule was small. However, the exercises assumed, in general, that the losses on the portfolios were smaller than what some independent central banks have experienced.

In this paper I complement the existing literature and I try to make progress understanding how a central bank that cannot rely on fiscal support *should* finance a temporary negative cash flow. Realistically, I assume that there is some uncertainty about the stochastic process for negative cash flow. I concentrate on the case in which the size of the negative cash flow is known but the duration of the negative cash flow phase is random. At the end of this period the net cash flow reverts back to zero. I find that the optimal policy is to choose an increasing sequence of nominal interest rates and to finance the shortfall with a changing mix of non-interest bearing liabilities (money) and interest bearing bonds (e.g. reverse repos). Initially most of the shortfall is financed by issuing bonds. As time goes by, if the negative cash flow continues, the optimal policy is to increase the nominal interest rate, print more money (which generates revenue) to fund both the interest payments on the debt and the negative cash flow.[4]

The basic intuition is related to the tax smoothing principle in Barro (1979). The setting is one in which there is a negative cash flow of random expected present value that has to be financed, and the only source of revenue is distortionary seigniorage. Thus, the problem is how to optimally choose the stochastic process for seigniorage. I find that, initially, seigniorage should be small and, as the negative cash flow period continues, it should be increasing. In all cases the amount of resources raised using a distortion causing tool falls short of the financing needs, including the transfer plus the interest on the debt.

Why not just picking a constant seigniorage as tax smoothing would imply? To understand the essence of the problem consider the case of a temporary transfer of known size and *duration*, that is, remove the uncertainty. Optimal smoothing

[4]In the section on extensions I discuss the case in which the cash flow turns positive. The qualitative results are similar.

considerations imply that distortions —in this case seigniorage— should be constant over time a constant level. Thus, the constant level of seigniorage is chosen so that its present value equals the present value of the negative cash flow.[5]

It follows that the modified version of distortion smoothing is the consequence of uncertainty. There is an option value associated with raising less than the full amount of revenue needed today (by issuing bonds) because if, say, the negative cash flow ends tomorrow then the optimal (constant from then on) level of seigniorage is whatever is needed to service the debt. Over time, during the negative cash flow phase, the size of the debt is increasing and, hence, the current seigniorage must increase as well. This is a consequence of optimal distortion smoothing.

The prescription of the optimal policy for interest rates (and inflation since the Fisher equation holds in this model) depends on the social cost of high inflations. Formally, I model high inflations as jumps in the price level. In the model I restrict the times at which the central bank can engineer these (costly) high inflation episodes. Jumps in the price level have the same effect as a default given that all liabilities are nominal. I find that the lower the cost of high inflations the lower the optimal interest rate (and inflation) during the pre-default time. The reason for this is simple: the possibility of (costly) default is equivalent to endowing the central bank with another "technology" to raise revenue. This implies a lower the reliance on seigniorage which, in turn, results in lower interest rates. The most constrained central bank faces an infinite cost of jumps in the price level and, consequently, follows what Auernheimer (1974) labeled as an "honest" policy: No jumps in the price level.

When allowing for gradual decreases in the transfer the model implies that the time path of interest rates and inflation is not monotonically decreasing. In the context of stabilization programs that specify decreasing monetization of deficits the results show that, under the optimal policy, inflation is not necessarily decreasing. Rather it can display a seesaw patterns with sharp decreases followed by prolongued increases.

The paper is organized as follows. The next section provides a brief (and incomplete) review of the literature on the impact of negative cash flows for the conduct of monetary policy. Section 3 presents the basic model. Section 4 describes the optimal policy in Phase II, that is, after the period of negative cash flow has ended, while section 5 studies the optimal policy in Phase I. Section 6 contains some extensions,

[5] Uribe (2016) shows that the optimal policy is to keep the inflation rate constant forever. In the case that the monetary authority can default at the time that net income returns to its normal level, Manuelli and Vizcaino (2017) show that inflation is high for as long as the central bank's net income is negative and it drops to zero when net income is normalized.

and section 7 offers some concluding comments.

2 Brief Literature Review

As the result of the 2008 financial crises, many central banks changed how they conducted monetary policy. Probably the most significant change was the expansion of their balance sheet. Central banks purchased large amounts of assets —many long dated securities— and financed the purchases issuing short term interest earning reserves. The difference in interest rates created an operating profit that was remitted to the Treasury. Even before the recent increases in nominal interest rates many observers noted that the situation could be reversed: if, for policy reasons, the rate paid by the central bank has to be increased while at the same time the income from its portfolio of long term assets does not change the monetary authority can potentially sustain operating losses.

In the case of the Fed, Goodfriend (2014) alerted to that possibility and argued that the issue of how to finance such a loss is a delicate and difficult problem.[6]

Del Negro and Sims (2015) describe a model that distinguishes between the central bank's balance sheet, with its own budget constraint, and the rest of the government budget. They study the conditions under which it is feasible to support a *given* policy that they model as a version of the interest smoothing version of the Taylor rule. They define fiscal support as a transfer from the Treasury to the central bank and show conditions under which a temporary negative cash flow *does not* require fiscal support to implement the given, arbitrary, policy. They parameterize a version of their model and argue that for the U.S. the Federal Reserve Bank is not likely to need fiscal support in the near future since they estimate that the present value of seigniorage is large. Crtical to their exercise is the assumption that the economy is *deterministic*. Thus, in the case Del Negro and Sims study, the right measure is to compare present values —using the risk free rate— of resources and liabilities. Alternative assumptions about the demand for money yield different estimates of the present value of resources available to the Fed (assets plus seigniorage) and, hence, potentially different views on the sustainability of a given policy in the absence of fiscal support.There are two major differences between Del Negro and Sims (2015) and this paper. First, they consider a time zero change in a variable —typically an

[6]Goodfriend considers the costs and benefits of financing this shortfall either issuing (non-interest) reserves or borrowing in the RRP market. He views both options as problematic and advocates instead building a buffer stock from earnings.

interest rate— that is perfectly predictable. Thus, they solve a non-stochastic model that was not designed to capture the challenges imposed by uncertainty over the size of the negative cash flow. Second, they study the feasibility of implementing a given class of monetary policy rules. By feasibility they correctly impose a transversality condition on the real value of central bank liabilities. In this paper, I am interested in understanding how should the central bank deal with this situation. In other words, what should the central bank do?

Hall and Reis (2015) also explore the sustainability of what they call a "dividend policy" which amounts to a rule that determines a transfer from the central bank to the treasury (which can be negative) as a function of the state of the economy. They emphasize that the "stability of a central bank depends crucially on what happens to its dividends when net income is negative." They study several policies that deal with strategies to manage the Fed's and the European Central Bank's balance sheet, and they emphasize different approaches to designing a financial policy with an emphasis toward what they view as a *financially stable policy* defined as a strategy that rules out explosive paths of interest paying liabilities issued by the central bank. They point out that they do not view their model as a useful way to study the inflationary consequences of negative net income. They state that "a small loss for the Fed would require very large increases in inflation." They claim that higher seigniorage is associated with a loss of central bank independence.

Hall and Reis (2015) and Del Negro and Sims (2015) emphasize that, given current institutional arrangements, it is important to recognize that the central bank and the Treasury have separate budget constraints when the Treasury commits to no transfers to the central bank. This case —which Del Negro and Sims (2015) labeled lack of fiscal support— clearly requires formally recognizing the connection between central bank decisions and fiscal policy. Cavallo et. al. (2019) discuss the fiscal implications of alternative strategies concerning the Fed's balance sheet normalization. Reis (2015) provides a summary.

Benigno and Nisticò (2015) describe a variety of neutrality and non-neutrality results associated with open market operations in a context where the central bank has a large balance sheet. They show that neutrality obtains only under relatively stringent conditions and that, if the central bank faces the risk of a negative net income it is likely that a traditional monetary policy will require fiscal support in the form of transfers from the treasury to the central bank.

Diaz-Gimenez et. al. (2008) show that time consistency limits the ability of the monetary authority to implement optimal policies. The key difference with this paper is that the path of deficits is not decreasing (and random), and that there is coordination between the fiscal and monetary authorities which implies that taxes other than the inflation tax are used to finance the service of the debt. Martin (2013) has a thorough discussion of optimal policies in economies with different frictions. He restricts himself to the case in which bonds have payoffs in real terms. In the context a the New Keynesian models, Leeper et. al. (2016) discuss optimal monetary and debt policy imposing time consistency. For a recent survey of monetary models and optimal policies see Canzoneri (2011). For a survey of central bank credit to governments see Jacome (2012).

3 The Economy

3.1 The Environment

I study the simplest model that allows me to capture the notion of a temporary negative cash flow of uncertain duration. Net income is given by $-P_t x_t$ where P_t is the price level and the process x_t is

$$x_t = \begin{cases} x & t \in [0, T] \\ 0 & t \geq T \end{cases} , \tag{1}$$

where $x > 0$.

T is an exponentially distributed random variable with parameter η. This implies that the expected duration of the negative net income phase is $1/\eta$. In section 5, I extend the model to allow for multiple phases of decreasing (in absolute value) net income. I label the random period of time in which $x_t > 0$ Phase I, while the period in which $x_t = 0$ Phase II.

There are at least two possible interpretations of x. In the case of inflationary finance, x is simply the amount that the central bank has to transfer to the Treasury. In the case of independent central banks it is the difference between what they earn on assets and what they pay for their liabilities.

I assume a pure exchange economy. Income per period is given by y, and consumption is $c = y - g$, where g is government spending. I simplify the real side of the economy so that I can concentrate on the optimal monetary policy. Extending the model to accommodate correlation between the negative net income period and real variables is part of ongoing research.

3.2 Households

I assume that there is a representative dynasty that has preferences over consumption and real money balances given by

$$U = E\left[\int_0^\infty e^{-\rho t}\left[u(c_t) + v(m_t)\right] dt\right],$$

where c_t is consumption at time t and $m_t = M_t/P_t$ is real money balances. The functions u and v are assumed strictly concave and twice continuously differentiable. The private sector budget constraint is given by

$$dM_t + dB_t^T + dB_t^M = \left(i_t(B_t^T + B_t^M) + P_t y_t - P_t c_t - P_t \tau_t\right) dt + d\tilde{N}_t, \tag{2}$$

where $\tilde{N}_t$ is the Poisson process that captures the (potential) jump in the price level associated with the end of the transfer period, that is, a potential jump that occurs at time T.[7]

On the income side, the representative dynasty earns interest at the nominal rate i_t on its holdings of treasury-issued debt, B_t^T, as well as the stock of monetary authority-issued debt, B_t^M.[8] In addition, the representative household earns income, spends resources purchasing consumption and pays taxes (if $\tau_t < 0$, the household receives a transfer). The notation emphasizes that it is possible to separate the policy decisions made by the treasury —choices of B_t^T and τ_t— from those made by the central bank.

It useful to consider what happens if the economy is in the first phase (that is $x_t = x$), and hence there are no jumps in the price level. In this case, the appropriate version of equation (2) is

$$dM_t + dB_t^T + dB_t^M = \left(i_t(B_t^T + B_t^M) + P_t y_t - P_t c_t - P_t \tau_t\right) dt,$$

where a dot denotes time derivative. Let W_t be total financial wealth. Thus,

$$W_t = B_t^T + B_t^M + M_t,$$

and with this notation the budget constraint in Phase I is

$$dW_t = \left(i_t W_t - i_t M_t + P_t y_t - P_t c_t - P_t \tau_t\right) dt.$$

[7] The assumption that money earns no interest is not essential. All that is required is that if i^m is the nominal return on money balances, then $i - i^m > 0$.

[8] If one views B_t^M as interest earning reserves, then money, M_t, is the set of liabilities that are used in transactions. For a transactions based model that derives a stable demand see Lucas and Nicolini (2013).

Let lower case letters denote real values of every variable. The real version of the budget constraint is then

$$dw_t = \left(\left(i_t - \pi_t \right) w_t - i_t m_t + y_t - c_t - \tau_t \right) dt.$$

What is the potential impact of the arrival of Phase II (at time T)? Since the only state variable is wealth, a jump in the price level can affect the real value of wealth. Let the post jump price level be denoted P_T' and the pre-jump simply $P_T = P_{T^-}$ which corresponds to the left limit of the price process. Real wealth before and after the jump is given by

$$w_T' = \frac{W_T}{P_T'}, \text{ and } w_{T^-} = \frac{W_T}{P_{T^-}},$$

This implies that

$$w_T' = w_T \frac{P_{T^-}}{P_T'}.$$

In order to avoid specifically using the (somewhat cumbersome) left limit notation I will denote

$$w_T = w', \; w_{T^-} = w, \; P_T = P' \text{ and } P_{T^-} = P.$$

The stochastic process for wealth is then

$$dw_t = \left(\left(i_t - \pi_t \right) w_t - i_t m_t + y_t - c_t - \tau_t \right) dt + \left(\left(i_t' - \pi_t' \right) w_t' - i_t' m_t' - \left(i_t - \pi_t \right) w_t + i_t m_t \right) dN_t. \tag{3}$$

, where a $'$ over a variable denotes its its value after the (possible) jump in the price level.

The problem faced by the representative consumer is simply

$$U = E\left[\int_0^\infty e^{-\rho t} \left[u(c_t) + v(m_t) \right] dt \right],$$

subject to equation (3), where the expectation is taken over the realization of the time of the first jump of the Poisson process.

Before analyzing the optimal monetary policy it is necessary to derive the optimal behavior on the part of the private sector, since their decision rules are constraints that the central bank has to incorporate to its optimal policy problem.

The HJB equation corresponding to the household's optimization problem is (ignoring the aggregate state that, in this setting coincides with the individual state)

$$\begin{aligned} \rho H(w) &= \max_{c,m} \left\{ u(c) + v(m) + H'(w) \left(\left(i - \pi \right) w - im + y - c - \tau \right) \right. \\ &\quad \left. + \eta \left[H(w \frac{P}{P'}) - H(w) \right] \right\}. \end{aligned}$$

The first order conditions are

$$u'(c) = H'(w),$$

and

$$v'(m) = H'(w)i.$$

Thus, the demand for money is simply given by

$$\frac{v'(m)}{u'(c)} = i. \tag{4}$$

Equation (4) completely summarizes optimal behavior on the part of the household. As such, it must be imposed as a constraint on the problem solved by the monetary authority.

If consumption is constant (as it will be the case in equilibrium) so is $H'(w)$. This implies that, in equilibrium, $H''(w) = 0$. Differentiating the HJB equation with respect to w and imposing the envelope condition we get that

$$(\rho + \eta) H'(w) = H'(w) (i - \pi) + \eta H'(w\frac{P}{P'})\frac{P}{P'}.$$

However, if equilibrium consumption is constant, it follows that

$$i = \rho + \eta (1 - P/P') + \pi,$$

and, hence, that the demand for money is

$$\frac{v'(m)}{u'(c)} = \rho + \eta (1 - P/P') + \pi. \tag{5}$$

In this economy the realized real interest rate *contingent on no arrival* of the Phase II (normal net income) is $r = \rho + \eta (1 - P/P')$. It follows that the *higher the expected jump in the price level the higher the realized real interest rate.* If, on the other hand, the optimal policy is restricted to continuous paths of the price level (that is, no default), then the real interest rate is equal to the discount factor. Moreover, it also follows that once the economy enters Phase II the real interest rate satisfies $r = \rho$.

3.3 Central Bank

To determine the optimal monetary policy it is necessary to be explicit about the objective of the central bank as well as the constraints that it faces. What is the monetary authority's budget constraint? Here, as the literature correctly emphasizes, it is important to separate the central bank's balance sheet from the consolidated government's budget constraint. The key difference is that, in this model, the central bank's only source of income that can be used to service any interest paying liability is seigniorage. It is possible, with some complication of notation, to add earnings from the assets in the central bank's portfolio to its budget constraint. In this case, the policy that I describe implies that whenever income is positive it is transferred to the treasury, while negative net income has to financed using seigniorage. The only qualitative difference with a model that allows for an explicit return on the portfolio is that the limit on central bank's liabilities must take into account the future earnings on its assets.[9] Thus, in what follows I use a simplified version of the central bank's budget constraint that embeds this transfer policy.

The central bank's budget constraint after time T, that is, in Phase II ($x_t = 0$) is

$$dM_t + dB_t^M = i_t B_t^M dt.$$

With a slight abuse of notation I use W_t to denote total liabilities of the central bank. $W_t = M_t + B_t^M$. The budget constraint is then given by

$$dW_t = \left(i_t W_t - i_t M_t\right) dt,$$

or, in real terms,

$$dw_t = \left(\left(i_t - \pi_t\right) w_t - i_t m_t\right) dt.$$

Since $i_t - \pi_t = r_t$ and $i_t = v'(m_t)/u'(c)$, then the Phase II budget constraint that incorporates optimal behavior by the private sector is

$$dw_t = \left(r_t w_t + x - \frac{v'(m_t)m_t}{u'(c)}\right) dt, \tag{6}$$

where r_t depends on the particular equilibrium that obtains.

[9]This possibility is discussed in the section on extensions. A more involved exercise would allow the central bank to accumulate precautionary reserves or to liquidate some good assets to make up the negative net income. However, I ignore this possibility since my objective is to determine how to deal with negative net income rather than how to prevent the possibility of negative cash flow.

It simplifies the presentation to rewrite the central bank's problem in terms of seigniorage. Let

$$z_t = \frac{v'(m_t)m_t}{u'(c)}.$$

I assume that seigniorage as a function of real money balances is single peaked and that the central bank operates on the efficient (downward sloping) side of the Laffer curve. In that case the previous condition defines m as a function of z. Let $L(z) \equiv v(m(z))$. Under standard conditions the function L is decreasing (higher seigniorage requires higher interest rates and lower real money balances) and concave.[10]

4 Optimal Policy: Phase II

Let time T be the realization of the random duration time corresponding to Phase I. The central bank is faced with a stock of liabilities, w_T and wants to maximize the utility of the representative agent. Since consumption is constant and there is no residual uncertainty after time T, the problem faced by the monetary authority is simply

$$\max_{z_t, w'} \int_0^\infty e^{-\rho t} L(z_t) dt - \kappa\Delta(w_T - w'),$$

subject to

$$dw_t = (\rho w_t - z_t)\, dt, \text{ with } w_0 = w' \text{ and } w' \leq w_T,$$

where the function $\kappa\Delta$ captures the notion of a default penalty. Higher values of κ correspond to costlier defaults. In particular, I take the case of the limit as $\kappa \to \infty$ as capturing the situation in which the price level does not jump. I assume that Δ is C^2 and defined on the nonnegative reals. It satisfies

$$\Delta(0) = 0,\ \Delta^{'}(x) > 0,\ \Delta^{''}(x) > 0,\ \text{and } \frac{(\Delta')^2}{\Delta''} \geq \Delta$$

[11]

[10] Concavity requires that the elasticity of the marginal utility of money balances be positive.

[11] This is satisfied, for example by $\Delta(x) = e^{\lambda x} - 1$

The Default Decision The default decision is made by the central bank at the time that the economy switches from Phase I to Phase II. Let z^+ be the maximum level of seigniorage that the central bank can raise. Since the real interest rate is constant and equal to ρ the highest level of central bank liabilities consistent with an equilibrium is given by

$$\rho w^+ = z^+ = z(m^+),$$

where m^+ is the level of real money balances that is associated with the highest level of seigniorage. If w_T is greater than w^+ the central bank must default. Of course it is possible that it would choose to default even if $w_T \leq w^+$.

Since in this phase all relevant quantities (z_t and w_t) are stationary, the objective function of the monetary authority is to choose δ —the post default level of liabilities— to maximize

$$\max_{\delta \leq min(w^D, w^+)} \left(\frac{L(\rho\delta)}{\rho} - \kappa\Delta(w - \delta) \right).$$

Let the solution be denoted $\delta(w)$. If $\delta(w) = w$ then it is optimal not to default. The highest level of central bank liabilities consistent with no default is given by

$$L^{'}(\rho w^D) + \kappa\Delta^{'}(0) = 0$$

provided that w^D is less than w^+. In the interior case, it follows that the higher the κ the higher w^D: Higher default costs enlarge the no default region.

Differentiation of the first order condition of the central bank's problem shows that $\delta^{'}(w) < 1$ and that $\delta(w)$ increases with κ. This implies, as expected, that the higher the cost of default the smaller the default.

The value of the central bank maximization problem at the start of Phase II (after the default decision has been made) is given by the C^1 function function M is given by

$$M(w) = \begin{cases} \frac{L(\rho w)}{\rho} & w \leq min(w^D, w^+) \\ \frac{L(\rho\delta(w))}{\rho} \text{-}\kappa\Delta(w - \delta(w)) & w \geq min(w^D, w^+) \end{cases}, \tag{7}$$

The optimal continuation policy is very simple:

- If the level of central bank liabilities is low at the time of the regime switch, that is, if $w_T \leq min(w^D, w^+)$ there is no default at T and, consequently, no large inflation..

- If $w_T > min(w^D, w^+)$ the optimal policy requires a partial default at T that brings the real value of the nominal liabilities of the central bank from w_T down to $\delta(w)$. This default is engineered by a jump in the price level. After the initial jump, inflation and real money balances are constant.

5 Optimal Policy: Phase I

In this section I study the optimal policy during the high negative cash flow phase (Phase I).

5.1 "Honest "Policy

I assume that the monetary authority is constrained to pick policies that rule out jumps in the price level.[12] This implies that $P' = P$ and that the real interest rate is constant and equal to $r = \rho$.

If the central bank pursues an honest policy this puts restrictions on how much debt it can issue during Phase I. In particular, the policy requires the debt level at the time that the economy switches to Phase two to be below the maximum, that, is $w_T \leq w^+$. To guarantee that this constraint holds there is a highest possible value of the central bank's liability during Phase I —which I denote w^1— that can not be exceeded. This value is given by

$$\rho w^1 + x = z(m^+) = z^+.$$

If $w > w^1$ then $dw > 0$ and it is unbounded. This implies that, for sufficiently long T, w_T will exceed the maximum feasible level of liabilities consistent with no default in Phase II (w^+). Thus, it is *not feasible* for an honest government to let the level of total liabilities exceed w^1 during the period in which net income is negative. This must be the case even though $w^1 < w^+$, that is, the maximum value of liabilities is strictly below the maximum sustainable without default.[13] If the central bank followed a policy that results in $w_t \in (w^1, w^+)$ at some time t, during Phase I, then such a policy has a non-zero probability of requiring a default when the economy switches to the normal, zero net income, phase.

[12]This can obtain as the limit when κ goes to ∞ in the cost of default function

[13]Note that w^1 is independent of any additional resources that the central bank can count on in Phase II. This, of course, is the consequence of market incompleteness.

Let $V(w)$ be the value of the central bank's problem when the state is w during Phase I. The HJB equation is[14]

$$\rho V(w) = \max_{z} \left\{ L(z) + V'(w)\left(\rho w + x - z\right) + \eta \left[M(w) - V(w)\right] \right\}, \tag{8}$$

where, in this case, $M(w)$ is given by the first branch of the function in equation (7).

The optimal choice of seigniorage satisfies

$$L^{'}(z) = V'(w). \tag{9}$$

The key properties of the optimal policy are summarized in the following proposition

Proposition 1 ("Honest" Optimal Policy) *Any optimal policy during Phase I is such that,*

- *If $w < w^1$, then z_t is increasing which implies that m_t is decreasing and i_t and π_t are increasing.*
- *The liabilities of the central bank, w_t are increasing and $w_t \to w^1$ (which implies $m_t \to m^+$).*

Proof. See Appendix. ■

The proposition characterizes the optimal monetary policy when net income is negative. For a given initial level of liabilities, w_0, the optimal policy picks an initial level of real money balances, m_0, and from then on the level of liabilities increase and real money balances are decreasing. This implies that the nominal interest rate given by

$$\frac{L(z_t)}{m_t} = \frac{v^{'}(m(z_t)}{u^{'}(c)} = i_t$$

is increasing. The inflation rate is also increasing and given by

$$\pi_t = i_t - \rho.$$

The economy displays the standard Fisherian result: Inflation and nominal interest rates move one for one.

[14] Existence of a solution follows from standard arguments. I assume that the solution is classical.

Along the optimal path total liabilities of the central bank are increasing. Since real money balances are decreasing, the ratio of bonds to money has an upward trend, and the central bank relies increasingly on debt issues to finance the negative cash flow. Thus, along the optimal path the central bank is actively engaged in open market operations that change the composition of its liabilities between non-interest bearing money and interest bearing bonds (or reverse repos).

Why is inflation increasing? When there is uncertainty about the size (or, equivalently, the duration of the transfer) there is an option value associated with initially picking a "too low" an inflation rate, in the sense that seigniorage falls short of financing needs. The reason is that the central bank chooses higher real money balances (and lower interest rates and a relatively smaller debt-money ratio) taking into account that there is a positive probability —constant in the Poisson setting— that the duration of Phase I will be short. As times goes by, the optimal policy relies on increasing debt issues as the best option to smooth out the distortion associated with non-constant real money balances. Under the honest policy there is an upper bound on the total liabilities of the central bank. The "honest" central bank will never exceed the limit w^1. At this point, inflation is at its revenue maximizing level.

At (random) time T the central bank engages in an open market operation that reverses the previous trend: Real money balances increase and nominal interest rates decrease as the ratio of interest earning debt to money decreases. There is a permanent drop in inflation (relative to Phase I). The inflation rate depends on the size of the central bank's liabilities at T (this, in turn, is a function of the duration of Phase I).

5.2 "Non-Honest" Policy

In this section I study the general case where the continuation value is $M(w)$ as given in equation (7). In this case the central bank will choose to default when the economy switches to Phase II if the stock of liabilities at that time is greater than or equal the minimum of w^D —the level at which it is optimal to default if given the possibility— and w^+ — the level of central bank liabilities that requires default. Thus, the effective default boundary is $min(w^D, w^+)$.

Even though the level of w^+ is independent of the features of the default penalty, the level of w^D is not. As argued above, w^D increases with the cost of default as parameterized by κ. For arbitrarily high values of κ —the case that captures the honest central bank optimal policy— there will be no default unless it is necessary. In that case $w^+ = min(w^D, w^+)$.

Proposition 2 ("Non-Honest"Optimal Policy) *The optimal policy in Phase I is characterized by*

- *For all levels of central bank liabilities, w, $z(w)$ is increasing in w, and the stock of debt is increasing over time ($dw > 0$). This implies that nominal interest rates and inflation are increasing as well*
- *If $w \leq min(w^D, w^+)$ then the real interest rate is $r_t = \rho$.*
- *If $w > min(w^D, w^+)$ then the real interest rate is $r_t = \rho + \eta\left(1 - \frac{\delta(w)}{w}\right)$. Since $\delta(w) < w$ the real interest rate is higher than in the no default region.*
- *Let $\kappa^{'} \geq \kappa$ and let $z(w, \kappa^{'})$ and $z(w, \kappa)$ be the optimal levels of seigniorage in each case. Then, $z(w, \kappa^{'}) \geq z(w, \kappa)$.*

Proof. See Appendix ■

In a qualitative sense, the optimal policy for a central bank that faces small default (or high inflation) costs is similar to the policy chosen by the "honest" central bank: Faced with a negative cash flow of uncertain duration the best policy is to increase the interest rate over time and to finance the shortfall borrowing from the private sector.

If the debt of the central bank is "small" there is no impact on real interest rates (nominal rates are increasing since the Fisher equation holds in this model) until the debt reaches the point where it is optimal (or necessary) to default. When the liabilities of the central bank reach that point, real interest rates increase to capture the potential loss of wealth associated with high inflations.

A "non-honest" central bank will in general choose higher debt levels than an "honest" central bank. In particular the liabilities of an "honest" central bank will never exceed $w^1 < w^+$. However, it is possible for a "non-honest" central bank to exceed the safe level and still not default if the switch to Phase II occurs before the liabilities reach the cutoff point.

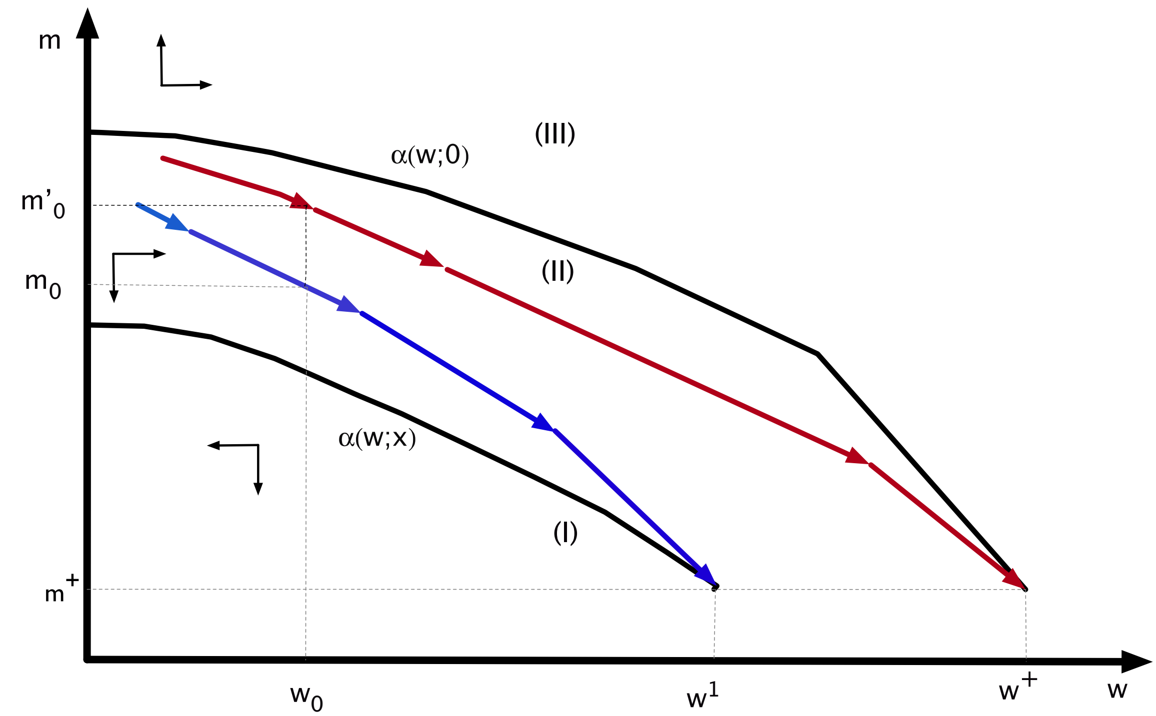

Figure 1: Optimal Policies. Blue ("Honest") and Red ("Non-Honest")

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Figure 1 displays two optimal loci in (w, m) space. The policy of increasing z and w corresponds to a decrease in real money balances, m.

The blue locus corresponds to the honest central bank. Along the equilibrium path the movement is in the southeast direction: liabilities are increasing and real money balances decreasing. For any level of liabilities, real money balances associated with the non-honest policy are larger than the corresponding level for the honest policy. This implies interest rates and inflation are lower.

6 Extensions

In this section I discuss some extensions of the simple model

6.1 Multiple Phases

How does the optimal policy change when the decrease (in absolute value) of the negative cash flow is gradual? Is it the case that under the optimal policy the nominal interest rate displays a monotonically decreasing path?

To discuss this case I allow for two phases of negative cash flow and I report the results in the case of an "honest" central bank. [15]

I restrict the presentation to just two levels of negative income but it is relatively easy to see how to extend the analysis to any finite number of rounds. As before, net income is $-x_t$ where x_t is given by

$$x_t = \begin{cases} x_0 & t \in [0, T_0] \\ x_1 & t \in [T_0, T_1], \\ 0 & t \geq T_1, \end{cases}$$

where $x_1 < x_0$ and, as before, I maintain the Poisson assumption with parameters η_0 and η_1. Thus, the environment is such that an initially high level of transfers (of uncertain duration) will be followed by a lower level (also of uncertain duration) and then a return to the normal (zero) net cash flow.

Let $V_j(w)$ be the central bank's value function when $x_t = x_j$. Then $V_1(w)$ coincides with the value function described in the case of the honest policy with just one level of x (if $x = x_1$). In each phase there is a highest level of liabilities consistent with no jumps in the price level. They are given by w_0^1 and w_1^1 given by

$$\rho w_0^1 = z(m^+) - x_0 < \rho w_1^1 = z(m^+) - x_1.$$

The relevant HJB equation is

$$\rho V_0(w) = \max_z \left\{ L(z) + V_0'(w) \left(\rho w + x_0 - z\right) + \eta \left[V_1(w) - V_0(w)\right] \right\}, \qquad (10)$$

The same arguments that we used before show that the function V_0 is concave, and that the level of liabilities is increasing over time. If it is C^2 taking the derivative on both sides of equation (11) one gets that

$$\eta \left[V_0'(w) - V_1'(w) \right] = V_0''(w) \left(\rho w + x_0 - z\right), \qquad (11)$$

[15] In this case there are two instances, the times T_0 and T_1 that, in principle are candidates for a default. This slightly complicates the analysis but does not add much to the stylized properties of optimal policies.

The right side of equation (11) is negative given the concavity of V_0 and that dw is positive. Thus,

$$V_1^{'}(w) = L^{'}(z_1(w)) \; < \; V_0^{'}(w) = L^{'}(z_0(w)),$$

which given concavity of the L function implies that $z_0(w)$ is greater than $z_1(w)$. Consequently, for the same level of liabilities, inflation is lower. However, there are paths for the time series of inflation that that display higher inflation after T_0 than the max before T_0. This can happen if, for example, if $T_0 \ll T_1$, and w_t is close to w_1^1. In this case inflation is at the highest level in the low negative cash flow stage.

$$m_t \to m^+ \text{ as } w_t \to w_1^1,$$

Figure 2 shows the possible paths of the central bank's liabilities, w, and the associated real money balances, $m(w)$.[16] The path in green corresponds to the initial phase, and the blue path to the second phase.

It also shows one possible realization (in dotted red) that assumes that the first phase ends when $w_0 = w_{T_0}$ and the second phase ends when $w_1 = w_{T_1}$.

[16] Recall that since the solution is on the efficient side of the Laffer curve, higher z corresponds to lower m and higher π.

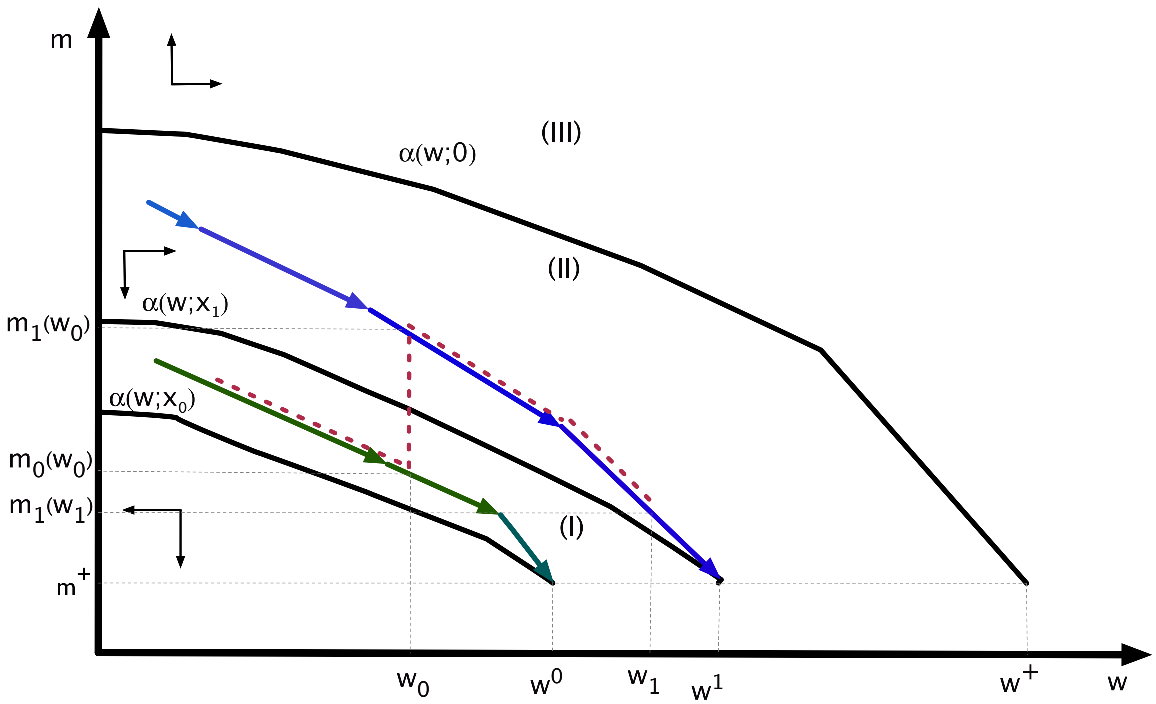

Figure 2: "Honest" Policy with Multiple Phases

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During the first phase real money balances decrease (and total liabilities increase) until time T_0. Right before the switch from the $-x_0$ to the $-x_1$ regime real money balances are $m_0(w_0)$. At $t = T_0$, the central bank —responding to the good news that the level of transfers has decreased— performs an open market operation that increases real money balances (decreases nominal interest rates) from $m_0(w_0)$ to $m_1(w_0)$. From then on, the path follows the blue line. At $t = T_1$, cash flow returns to normal (zero). In Figure 2 w_{T_1} is denoted w_1. At that point, The new level of real money balances (not shown) is greater than $m_1(w_1)$, and inflation is permanently lower.

It is interesting to note that even though net income is in a very clear sense weakly increasing (transfers are weakly decreasing) inflation and interest rates do not display a monotone path. Figure 3 shows that time path of inflation (nominal interest rates are simply $\rho + \pi_t$).

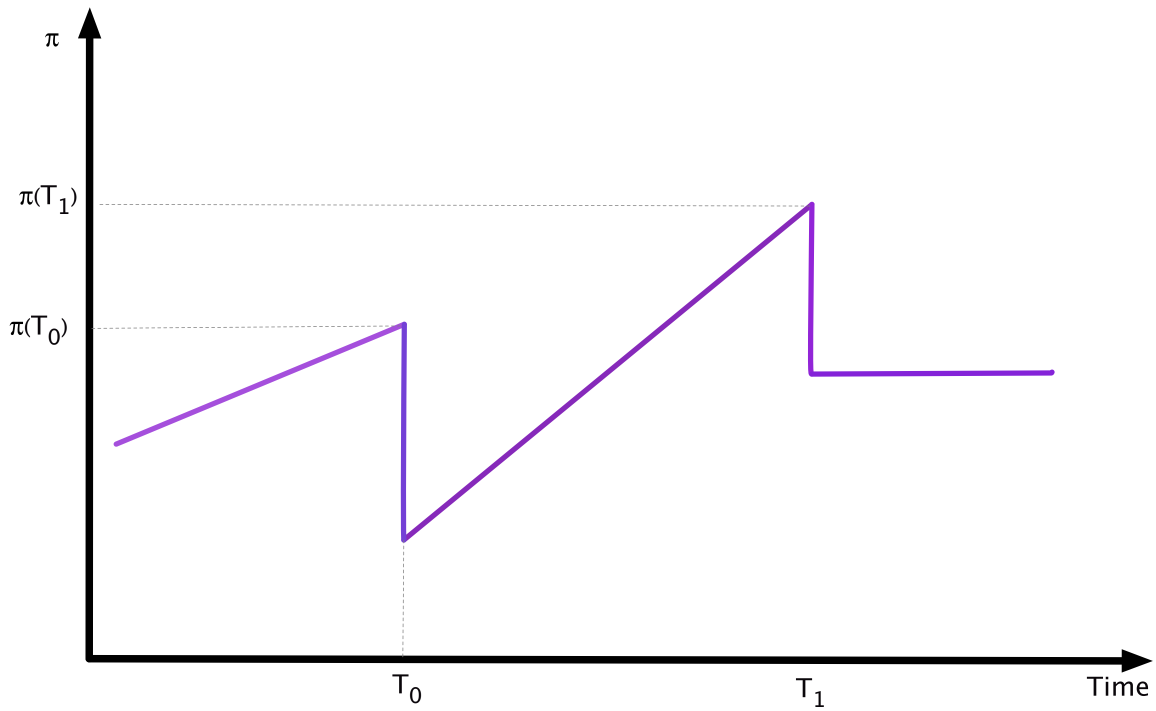

Figure 3: Time Path of Inflation

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As in the case of a single stage, the optimal policy initially displays low nominal interest rates and low inflation. As time goes by and the central bank finds itself in the need to keep making transfers, it chooses a higher nominal interest policy as well as a higher ratio of interest bearing liabilities to non-interest bearing money.

At the time that "good news" arrive in the form of higher (less negative) net income, the central bank adjusts its portfolio discretely: it repurchases some bonds by issuing money. Alternatively, this action can be described as lowering the nominal interest rate that decreases the demand for interest bearing debt and increases the demand for non-interest bearing money. After the decrease in the nominal interest rate the central bank keeps increasing the nominal yield on its interest bearing liabilities and accommodates higher inflation. This pattern continues until time T_1. At the point the net income is zero and the central bank picks a nominal interest rate that is consistent with raising sufficient seigniorage to finance the interest payments on the interest earning liabilities.

6.2 Optimal Unpleasant Monetarist Arithmetic

The basic model assumes that when the economy switches to Phase II (the zero net cash flow) the level of liabilities of the central bank cannot exceed what can be financed by the maximum level of seigniorage, w^+. This choice is, in some sense, arbitrary. If the monetary authority decided that the maximum long run inflation associated with w^+ is excessive, it can set a lower limit, sat $w^* < w^+$ not to be exceeded in Phase I.

What are the consequences? Put simply, the lower debt limit results in higher inflation in Phase I.

To see this, let $V(w; w^*)$ be the value function of the problem when the debt limit is w^*. Then totally differentiating the HJB equation with respect to w^* and imposing the envelope condition one gets that

$$(\rho + \eta)\frac{\partial V(w; w^*)}{\partial w^*} = \frac{\partial^2 V(w; w^*)}{\partial w \partial w^*}\left[\rho w + x - z\right]$$

Since increasing w^* relaxes the constraints faced by the central bank it follows that $\partial V(w; w^*)/\partial w^* > 0$. The same argument as in Proposition 1 shows that $\rho w + x - z$ is positive. Thus, $\partial^2 V(w; w^*)/\partial w \partial w^* > 0$. Since

$$L^{'}(z(w; w^*) = \frac{\partial V(w; w^*)}{\partial w}$$

and $L^{''} < 0$ differentiating both sides with respect to w^* proves the result.

Under the optimal policy there is no free lunch: the price of lower inflation today is higher inflation tomorrow. In particular governments that revise their long run inflation targets —in an MIT shock type of shock— in the direction of accepting higher inflation (and higher nominal interest rates) can implement a policy of lower interest rates (and lower inflation) in Phase I.

In this model it is trivial to show that the optimal w^* is w^+. The argument is simple: Choosing any level of maximum liabilities below the highest feasible adds a constraint to the problem and, hence, lowers the value.

6.3 A Central Bank with Resources

I have assumed so far the only genuine source of income for the central bank is seigniorage. This is —at least in the case of many developed countries central banks— a rather extreme assumption. Given their monopoly power, central banks can issue low yield securities to acquire assets that earn higher returns. This can give rise to a surplus.

In the case of the Fed, this surplus is typically rebated to the Treasury as a form of dividend. In this section I show that central banks that have access to resources in Phase II will choose to raise less seigniorage in Phase I but that the impact is limited.

To simplify, I will just specify that, in Phase II, the central bank receives real (flow income) equal to ρb. (In general this depends on monetary policy.)

I will also assume a sort of lexicographic preference (equivalently a constraint on the optimal policy) to address the question of sustainability of a given Taylor rule. I assume that the central bank follows the optimal policy unless this calls for negative inflation. Thus, the central bank will restrict itself to nominal interest rates $i \geq \rho$.

The budget constraint is

$$dw = (\rho w - \rho b - z + \tau)\, dt.$$

where τ is a non-negative transfer to the Treasury. Let z^* be the level of seigniorage corresponding to $i = \rho$.

Since it is optimal to smooth out distortions in this stationary environment the optimal continuation policy (In Phase II) requires that $dw = 0$. This implies that

$$\tau = \rho(b - w) + z.$$

where

$$z = max(z^*, \rho(w - b).$$

If in Phase II the level of liabilities falls short of the level of income, $b - w > 0$, the optimal policy is to set $i = \rho$ and to remit $\rho(b - w) + z$ to the Treasury.

If $z^*/\rho + b$ falls short of w then the interest rate will be set higher (to generate seigniorage) and the Treasury gets no dividend.

If in the base case (that assumed $b = 0$) the initial interest rate is below ρ, then the constraint on the nominal interest rate will be binding until the liabilities of the central bank reach some level $\bar{w} < w^1$ that depends on w_0 (it could be equal to w_0), as well as x and b, such that the optimal policy in Phase I is given by

$$z(w; b) = \begin{cases} z^* & \text{if } w \leq \bar{w} \\ z(w; b) > z^* & \text{if } w > \bar{w} \end{cases}$$

and $z(w, b)$ is increasing in w.

If $w > \bar{w}$ and $b' > b$,

$$z(w; b') \leq z(w; b)$$

with the inequality strict if the optimal level exceeds z^*.

This impact of "future wealth", b is limited: During Phase I w_t is bounded above by w^1, which is independent of b. What about Phase II? If $b \geq w^1$ then the monetary authority chooses $i = \rho$.. Thus a sufficiently large income in the second phase guarantees that the central bank can implement its preferred interest rule.

However, no matter how large b is, if Phase I lasts for sufficiently long time (or if x is very large) the central bank will deviate from the preferred monetary rule during Phase I.

It is possible to use this scenario and ask a question reminiscent of the question asked by Del Negro and Sims: What is the $Prob\left[T \leq T^*\right]$, where

$$T^* = \frac{1}{\rho} ln \left[\frac{\rho w_0 + x - z^*}{\rho \bar{w} + x - z^*} \right]$$

I take that the message form this section is that the **qualitative** properties of this "modified" rule coincide with the results in the basic model: If the duration of Phase I is long enough, the central bank will be required to raise seigniorage and, consequently, to increase nominal interest rates.

7 Concluding Comments

What are the lessons for the central banks that could find themselves in a situation in which net income is temporarily negative (deficit) and that there is no forthcoming fiscal support? The first lesson is that having the possibility to issue non-interest earning money (or reserves)[17] as well as other liabilities that earn the market rate of return —reverse repos in the case of the Fed and also some proposals sympathetic to the possibility that the ECB will issue Eurobonds (see, for example, Corsetti et. al. (2016))— is an essential component of the optimal policy mix.

The results show that there is no simple one to one correspondence between the negative cash flows and interest rates and that it is always optimal (in Phase I) to collect a level of seigniorage that falls short of the cash flow plus interest on existing debt. Thus, during the period of deficits the optimal policy call for continuing (and increasing) indebtedness by the central bank.

The intuition for this result is that uncertainty about the duration of the negative cash flow phase creates an option value for keeping inflation low (raising less rev-

[17]As discussed before all that is necessary is that this special liability earn less than the market return.

enue than necessary and borrowing the difference) since this allows for and optimal smoothing of the distortions associated with raising seigniorage.

The optimal policy viewed as an interest rate policy does not resemble a Taylor rule. In the case of the honest policy (no surprise inflation) the nominal interest rate is an increasing function of the real value of the total liabilities of the central bank. An outside observer just looking at the data will find a positive relationship between inflation and nominal interest rates. In the case of the honest policy the model implies that $i = \rho + \pi$ so that the coefficient on the observed policy rule is one.

One interesting finding —mostly in the context of stabilization plans— is that gradual decreases in central bank financed transfers need not be accompanied by steady decreases in inflation. There are realizations of the relevant stochastic process (the timing of reductions) that display a sew-saw pattern of inflation over time.

The "non-honest" optimal policy —one that allows the central bank to engineer large, surprise, inflations— requires a lower, relative to the honest policy, nominal interest rate and a larger level of maximum central bank debt. Any shock that lowers the social cost of default is associated with decreases in current inflation at the expense of possible larger jumps in the price level when Phase II arrives.

Finally, a central bank that has access to other forms of genuine resources in Phase II may be able to follow its preferred monetary rule during early stages of Phase I. In this case there is no conflict between current (potentially non optimal) and optimal policies. However, if this phase lasts a sufficiently long period, then the optimal policy will require that interest rates increase and additional seigniorage is raised.

8 Appendix

8.1 An alternative Formulation of the Penalty Function

Assume that the cost of default is a disruption of production which results in consumption being temporarily lower.

Let post default consumption be $y - \theta\Gamma(w - \delta(w))$, and assume that with expected duration $1/\alpha$ the economy goes back to normal.

The continuation utility of the central bank is then

$$\frac{1}{\rho}\left(u(y) + L(z(\delta))\right) + \frac{1}{\rho+\alpha}\left[u(y - \theta\Gamma(w-\delta)) - u(y)\right]$$

The term

$$\frac{1}{\rho+\alpha}\left[u(y - \theta\Gamma(w-\delta)) - u(y)\right],$$

corresponds to

$$-\kappa\Delta(w-\delta)$$

In this case the real interest rate during Phase I has to be adjusted to take into account the effect of the loss of consumption. The appropriate rate is

$$r = \rho + \eta\left[1 - \frac{u'(y - \theta\Gamma(w-\delta))}{u'(y)}\frac{\delta(w)}{w}\right]$$

Since

$$\frac{u'(y - \theta\Gamma(w-\delta))}{u'(y)} > 1$$

the real interest rate is lower than in the base case (for a given level of default).

8.2 Proofs[18]

Proof. Proposition 1. The optimal policy problem is The HJB equation corresponding to the optimal policy in Phase I under the restriction to a continuous path of the price level is

$$V(w) = \max_{z_t} E\left[\int_0^T e^{-\rho t} L(z_t) dt + e^{-\rho T} M(w_T)\right],$$

subject to

$$dw_t = (\rho w_t + x - z_t)\, dt, \text{ with } w_0 = w$$

where the expectation is taken over the realization of T.

Since $L(z)$ is concave and the constraint linear the function $V(w)$ is concave.

The HJB equation associated with this problem is

$$\rho V(w) = \max_z \{L(z) + V'(w)\,(\rho w + x - z) + \eta\,[M(w) - V(w)]\}$$

where, in this case, $M(w)$ is given by the first branch of the function in equation (7).

The first order condition defines $z(w)$ by

$$L^{'}(z(w)) = V'(w) < 0. \tag{12}$$

Concavity of V and L implies that $z(w)$ is increasing in w.

To show that z is increasing in x, totally differentiate the HJB equation with respect to x and impose the optimality condition to get that

$$\eta \frac{\partial V}{\partial x} = \frac{\partial V^{'}}{\partial x} < 0$$

since increases in x lower utility. From

$$L'(z(w;x) = V'(w;x)$$

it follows that $\partial z/\partial x > 0$.

To show that z is decreasing in η, let $V(w;\eta)$ be the value of the problem. Assume that $\eta' > \eta$, then the HJB equation implies that

$$\begin{aligned}(\rho+\eta')V(w;\eta') - \eta' M(w) &= L(z(w;\eta') + V'(w;\eta')[\rho w + x - z(w;\eta')] \\ &\geq L(z(w;\eta) + V'(w;\eta')[\rho w + x - z(w;\eta)]\end{aligned}$$

[18] These are "lazy" proofs that assume more differentiability than needed. The arguments go through by carefully taking differences and imposing the right bounds

If, contrary to the claim $z(w;\eta') > z(w;\eta)$ this implies that $V'(w;\eta')$. This in turn, requires that $(\rho+\eta)V(w;\eta) - \eta M(w)$ be a decreasing function of η. However, since $V(w;\eta)$ is increasing in η this is a contradiction.

Since $dw > 0$, then it either converges to some $\tilde{w} < w^1$ or it converges to w^1. Consider the value of the program at $\tilde{w}$. Since $dw = 0$ it follows that

$$V(\tilde{w}) = \frac{\eta M(\tilde{w}) + L(\rho\tilde{w} + x)}{\rho + \eta}$$

Since

$$M(w) = \frac{L(\rho w)}{\rho} \text{ it follows that } M'(w) = L'(\rho w)$$

Differentiating the $V(\tilde{w})$ function it follows that

$$V'(\tilde{w}) = \frac{\eta L'(\rho\tilde{w}) + \rho L'(\rho\tilde{w} + x)}{\rho + \eta} > L'(\rho\tilde{w} + x)$$

which implies that $z(\tilde{w}) = \rho\tilde{w} + x$ (as required by the convergence assumption) violates the first order condition. Since $V'(\tilde{w}) > L'(\rho w + x)$, the optimal z is strictly less than $\rho w + x$ which implies that, at $w = \tilde{w}$, $dw > 0$. ■

Proof. Proposition 2 The arguments about the properties of the $z(w)$ function as a function of x and η are the same as in the proof of Proposition 1. The arguments to show that

$$z(w) < r_t w + x \text{ and } z(w') \geq z(w) \text{ if } w' \geq w$$

also follow exactly the same derivations as in the proof of Proposition 1. Finally, the behavior of interest rates as a function of w follows from the household's first order conditions.

The new element is the impact of κ on seigniorage. Given that

$$L'(z(w;\kappa)) = \frac{\partial V(w;\kappa)}{\partial w}.$$

Showing that $z(w;\kappa)$ is increasing in κ, is equivalent to showing that the marginal cost of central bank liabilities, $\partial V(w;\kappa)/\partial w$ is increasing with respect to κ.

To establish the results is useful to recall that the HJB equation —where the dependence on κ is made explicit— is

$$\rho V(w;\kappa) = \max_z \{L(z) + V'(w;\kappa)(\rho w + x - z) + \eta [M(w;\kappa) - V(w;\kappa)]\}$$

where, in this case, $M(w;\kappa)$ is

$$M(w;\kappa) = max_{\delta}\left[\frac{L(\rho\delta)}{\rho} - \kappa\Delta(w-\delta)\right].$$

The maximization determines the optimal default function $\delta(w;\kappa)$. Simple calculations show that

$$\frac{\partial\delta}{\partial\kappa} = \frac{\Delta'}{\kappa\Delta'' - L''(\rho\delta)}$$

and

$$\frac{\partial\delta}{\partial w} = \frac{\kappa\Delta''}{\kappa\Delta'' - L''(\rho\delta)}.$$

Totally differentiating the HJB equation with respect to κ implies that a sufficient condition for $\partial^2 V/(\partial w \partial\kappa) \geq 0$ is

$$\frac{\partial V}{\partial w}\frac{\partial\delta}{\partial\kappa} + \Delta \leq 0$$

since $\partial V/\partial\kappa < 0$.

Totally differentiating the HJB equation with respect to w it follows that concavity (with respect to w) implies that

$$\frac{\partial V}{\partial w}\frac{\partial\delta}{\partial w} + \Delta' \leq 0$$

Using the inequality implied by this last condition on the previous one, it follows that

$$\frac{(\Delta')^2}{\kappa\Delta''} \geq \Delta$$

implies $\partial^2 V/(\partial w \partial\kappa) \geq 0$, which completes the proof.

■

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