Measurement and Stylized Facts in Growth and Development

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Data sources: Acemoglus (2009), Aghion-Howitt (2009), Jones (1998, 2015)

B. The Big Picture

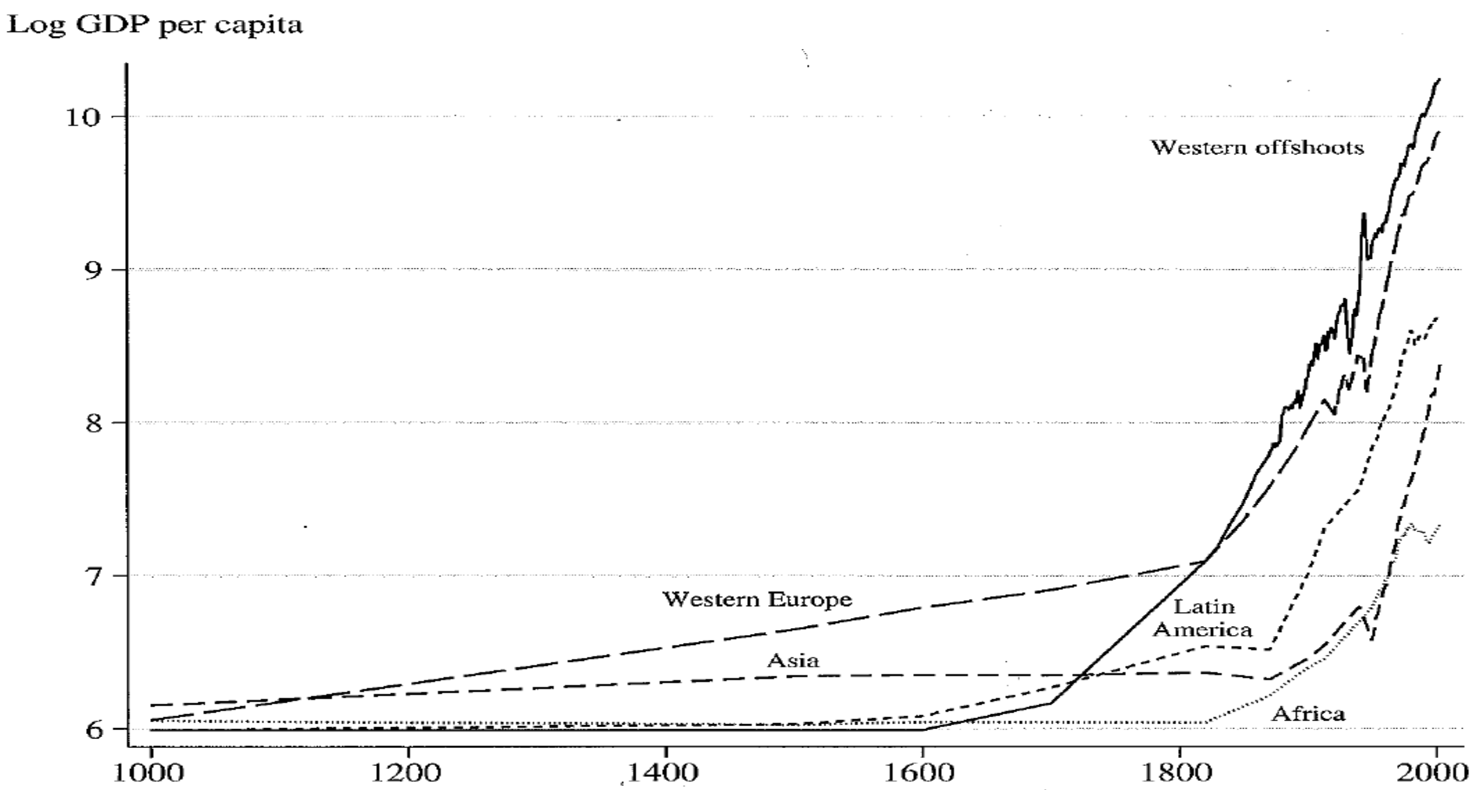
1. Long Term Development of the World Economy

- **Five ancient empires:**

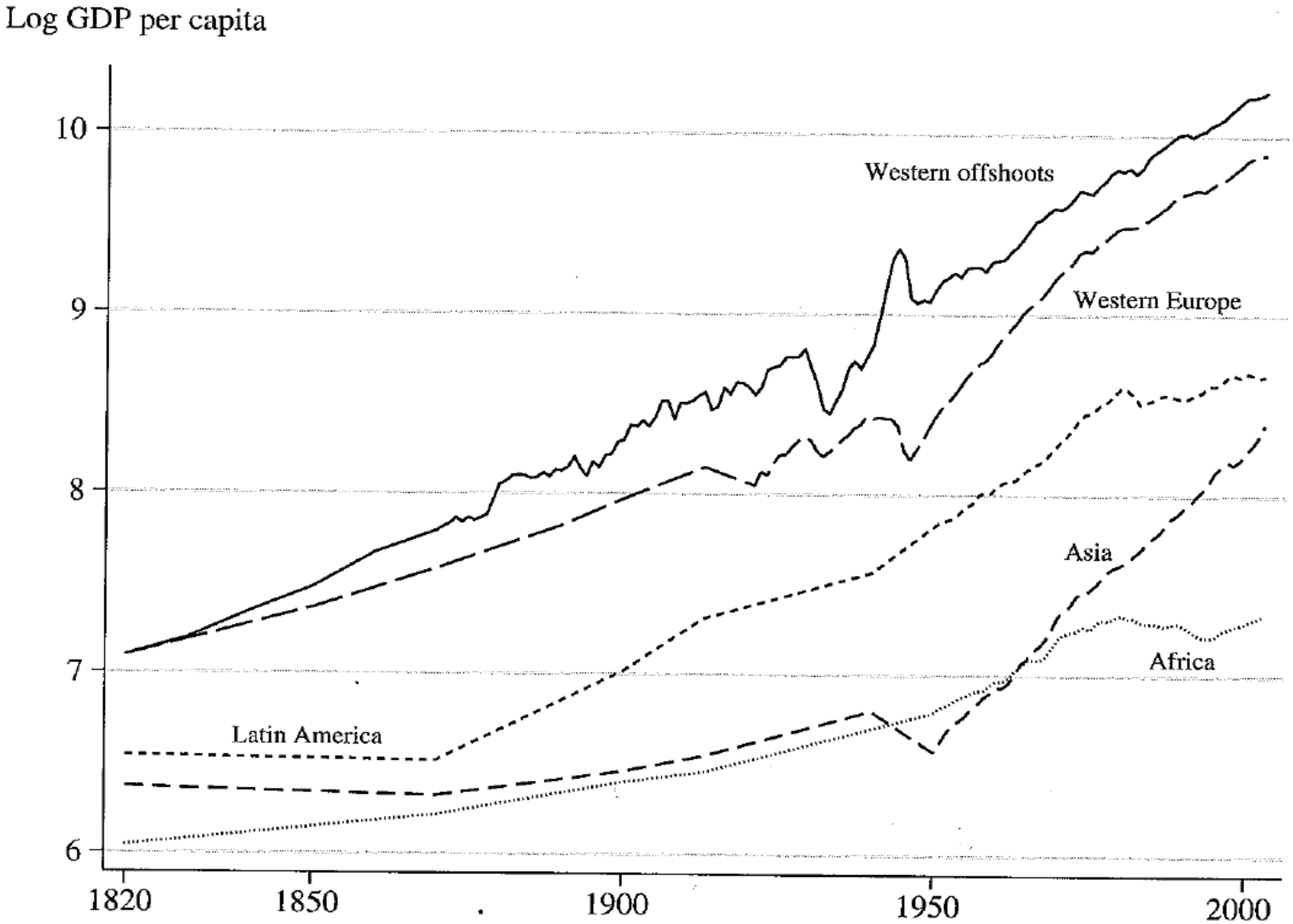
Greek empire (2000-300BC)		Chinese empire (2852BC-1911)
	Babylonian empire (1696-539BC)	
Egyptian empire (4000-30BC)		Indian empire (3300BC-1818)

- **The rise of Europe:**
  - **Roman empire (27BC-1461)**
  - **Spanish empire (1519-1898)**
  - **Dutch empire (1579-1795)**
  - **British empire (1689-1997)**
  - **German empire (1871-1918)**
- **The rise of America (1776-now)**

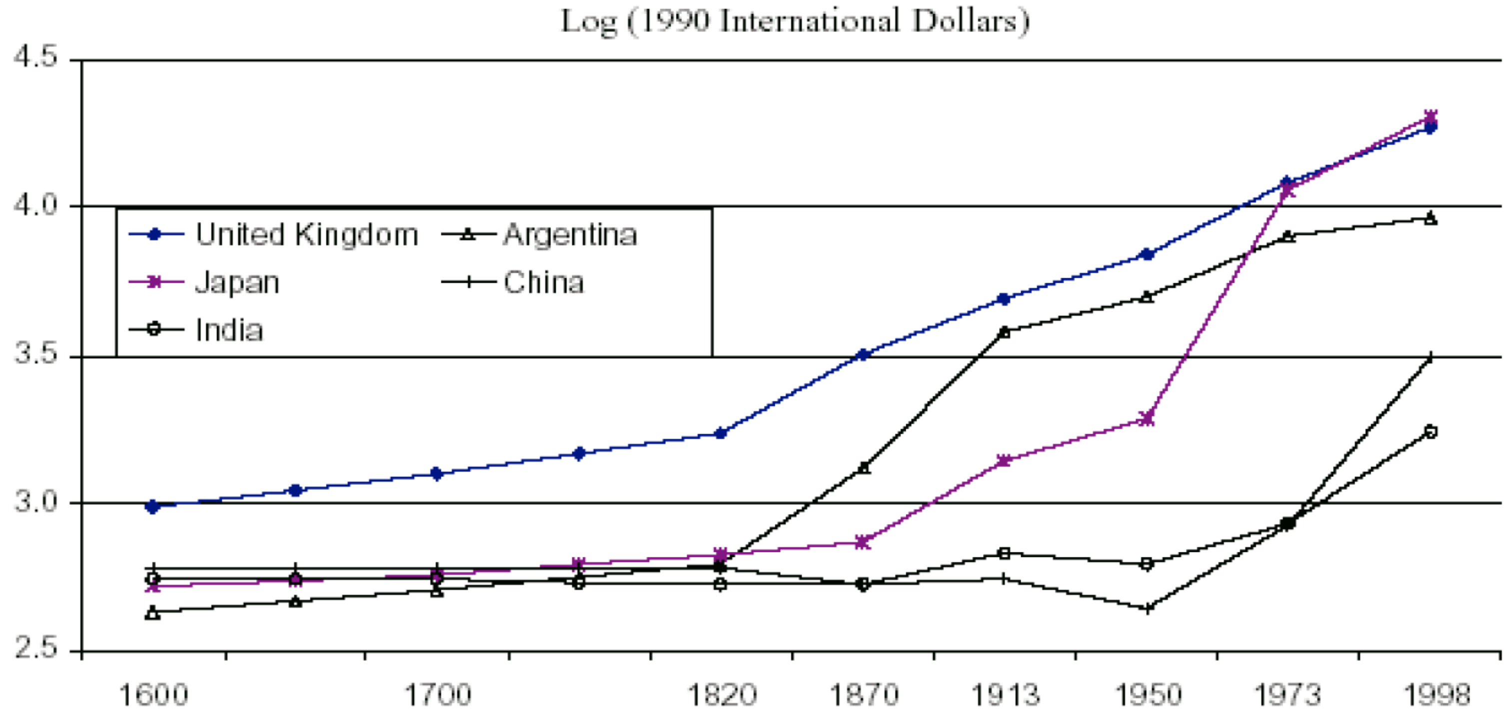
- **World development since 1000: Overtaking of Western Offshoots**
  - **Maddison data (scarce prior to 1820)**
  - **Western offshoots (former colonies of Western Europe)**
  - **Asia (historically China + India)**



- **World development since mid-19th century:**

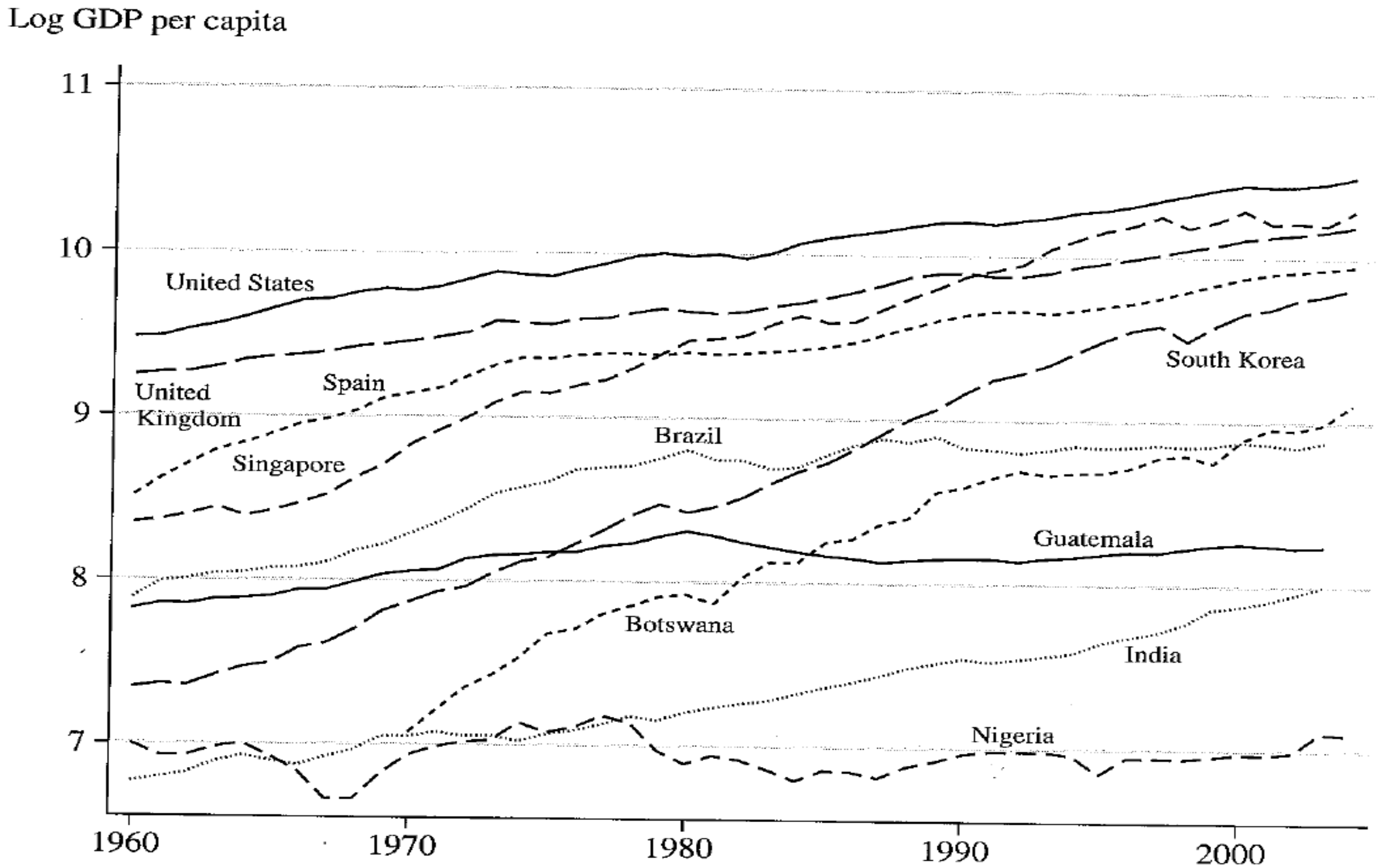


- **A closer look at 5 representative economies since 1600 (Ngai 2004):**



  - **Sustainable growth in real income per capita only started 1780**
  - **Argentina: post-independent federation since 1861**
  - **Japan: Meiji Restoration in 1868; lost decade(s) since 1991**
  - **China: wars; great leap forward; cultural revolution; post-1979 open-door policy (market/trade) + 1992 Southern Tour (FDI)**
  - **India: 1980's reform by Indira Gandhi; 1990's reform by Manmohan Singh**

2. The Post-WWII Era

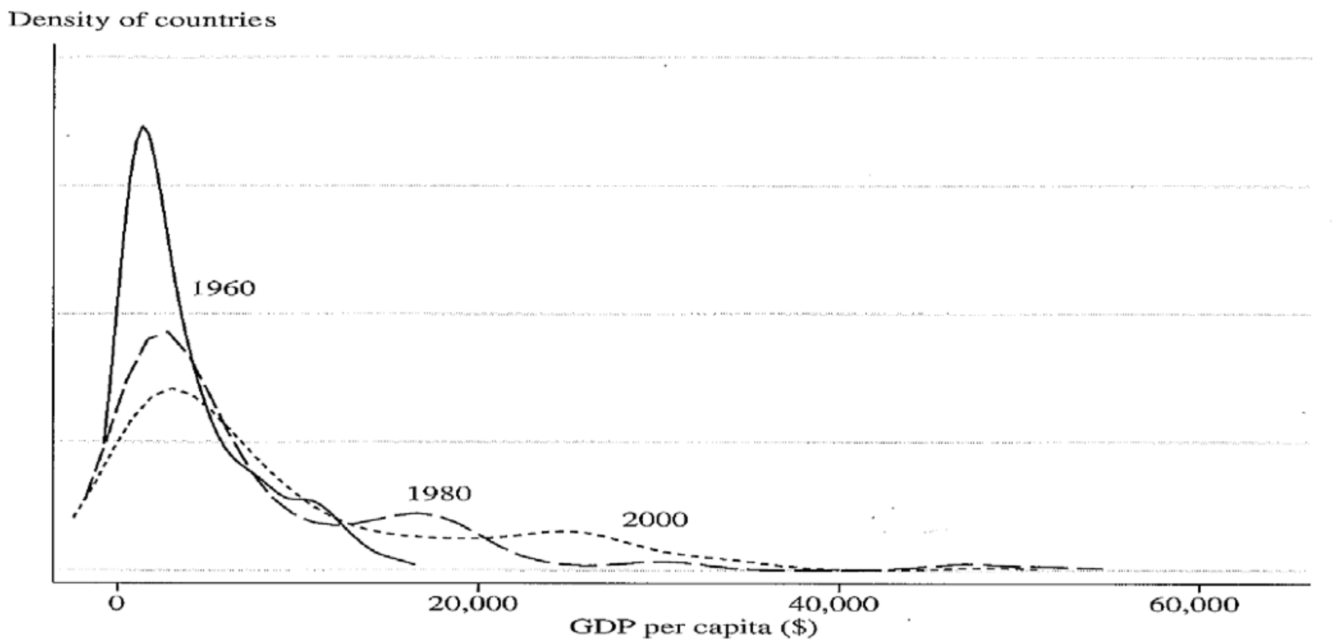


- **East Asian Miracles (Asian Tigers)**
- **African Miracle (Botswana)**
- **Poverty Traps (Nigeria and many Sub-Saharan)**

B. Cross-Country Study of Economic Growth

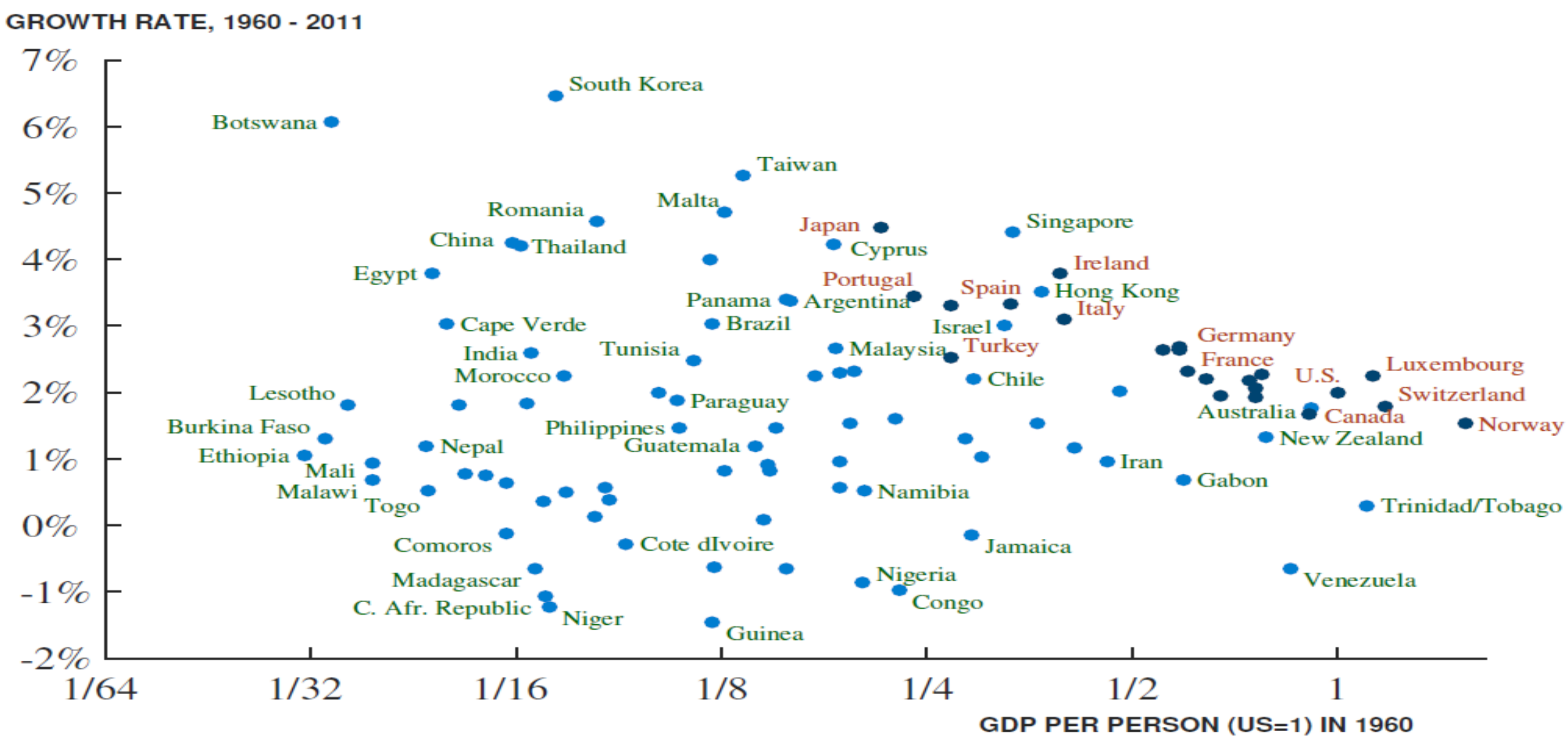
1. Overview

- **Distribution of world real GDP per capita**



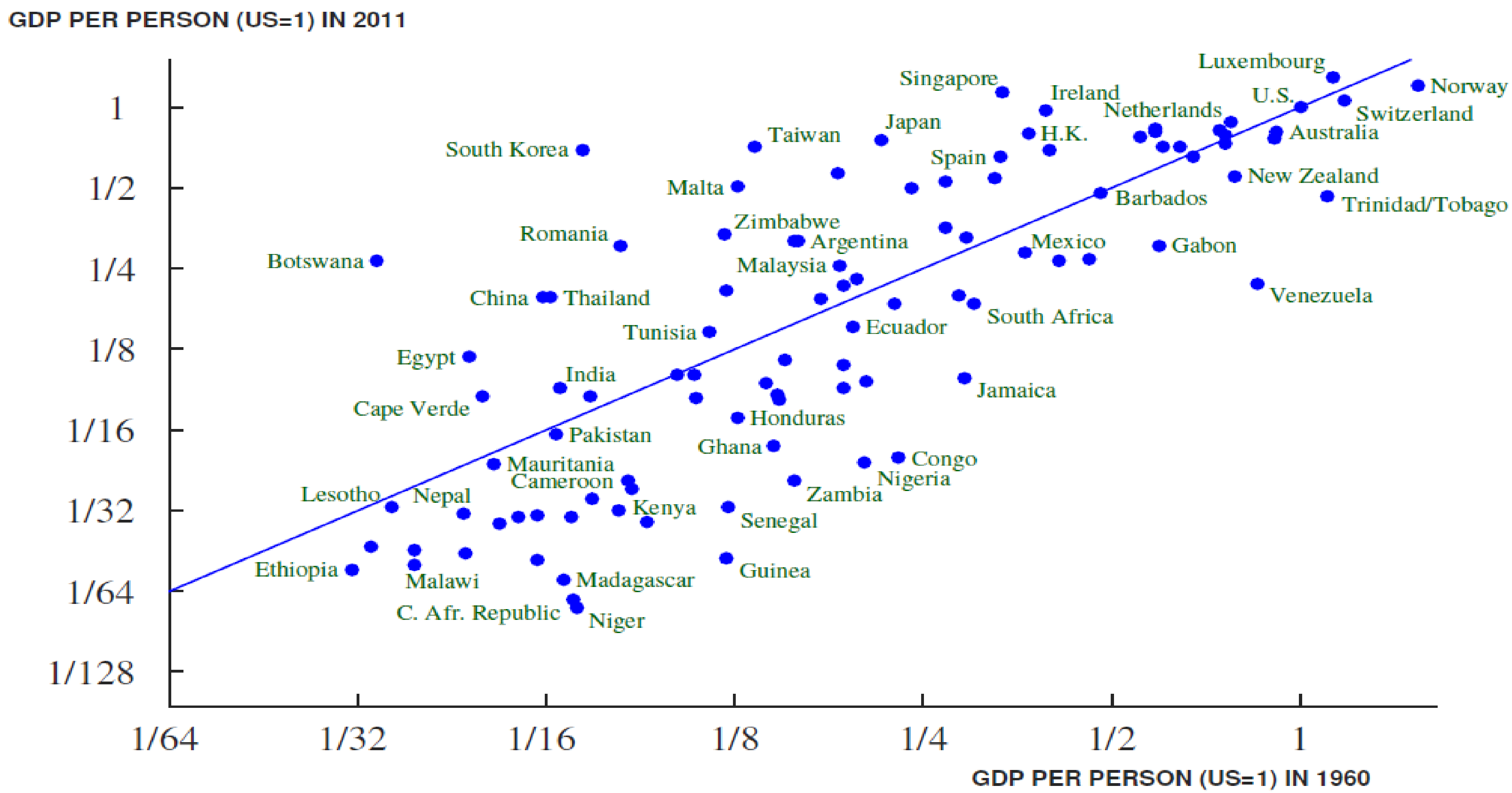
- **Widened world income distribution (cross-country inequalities)**
- **Rightward shift (upward economic development)**
- **Twin Peaks (Quah): poverty + middle income traps**

- **Cross-country growth experiences compared (relative income to US)**



- o **OECD frontier**
- o **Asian miracles**
- o **Development laggards: high initial relative income/low growth**
- o **Poverty traps (1/10 of US): Sub-Saharan & others**
- o **Convergence of the first 2 groups**

- **Cross-country income mobility:**



- Great divergence:
  - **Upward/downward mobility: miracles/laggards**
  - **Persistently high/low: developed/poverty traps**

- **Growth miracles and disasters (1969-1990)**

Miracles	Growth	Disasters	Growth
Korea	6.1	Ghana	–0.3
Botswana	5.9	Venezuela	–0.5
Hong Kong	5.8	Mozambique	–0.7
Taiwan	5.8	Nicaragua	–0.7
Singapore	5.4	Mauritania	–0.8
Japan	5.2	Zambia	–0.8
Malta	4.8	Mali	–1.0
Cyprus	4.4	Madagascar	–1.3
Seychelles	4.4	Chad	–1.7
Lesotho	4.4	Guyana	–2.1

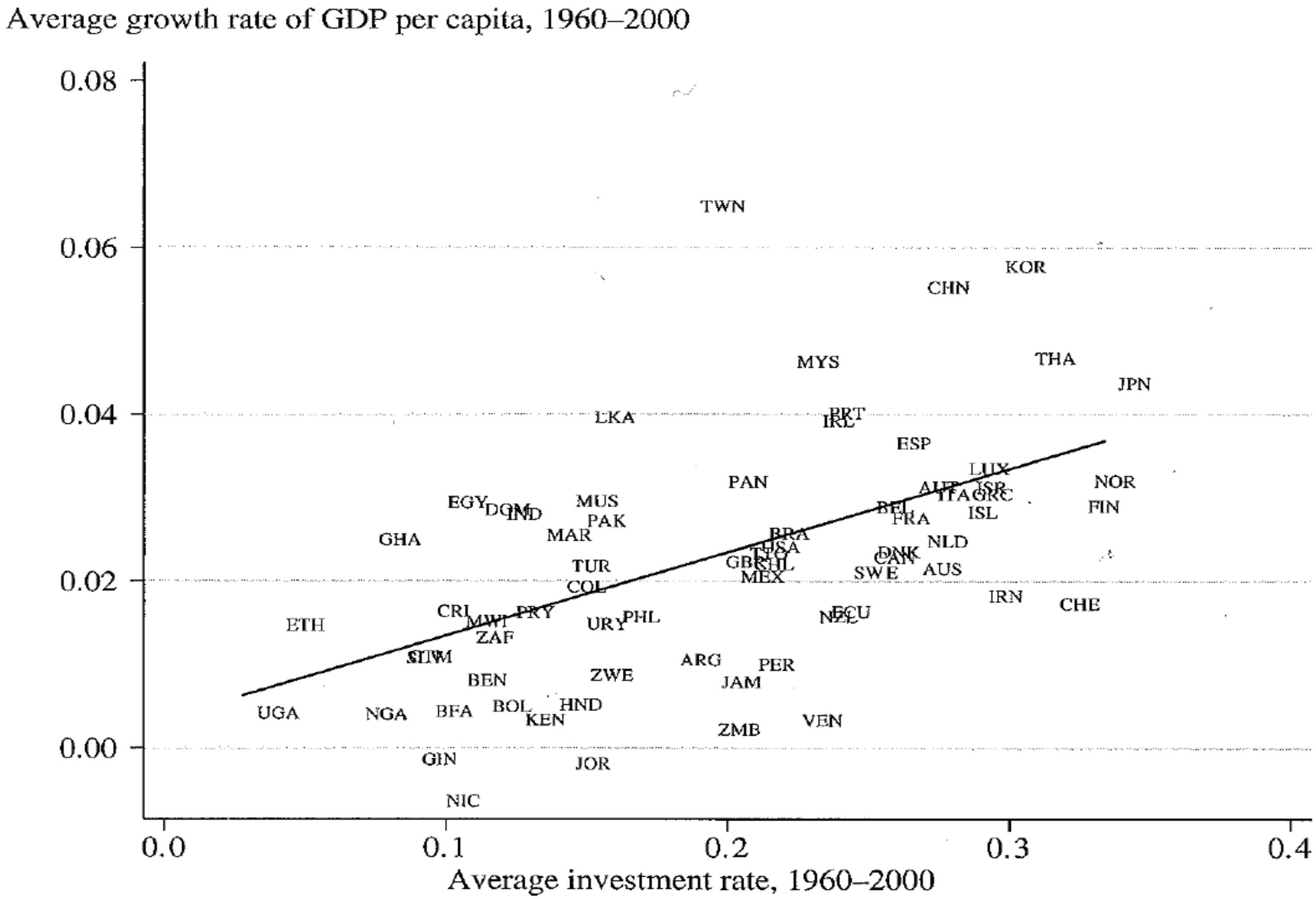
2. Determinants of Economic Growth:

- **Using neoclassical theory as an organizing framework, Jones-Manuelli (1997), Boldrin-Chen-Wang (2004) and Jones (2015) provided comprehensive surveys on the sources of per capita real GDP growth**
- **The determinants of economic growth:**
  - **Organizing framework: aggregate production: Y = A*F(K,H*L)**
    - **K: physical capital accumulation: saving, investment**
    - **L: labor force growth (labor participation); population growth – a negative factor (fertility choice)**
    - **H: human capital enhancement: education (years of schooling), learning by doing, job training**
    - **A: total factor productivity (TFP): R&D and technology invention, imitation, and adoption**
  - **other factors:**
    - **trade (final goods, intermediate goods)**
    - **institutions/infrastructures**
    - **finance/geography/urbanization**
    - **policy (monetary, fiscal, patent, population, others)**

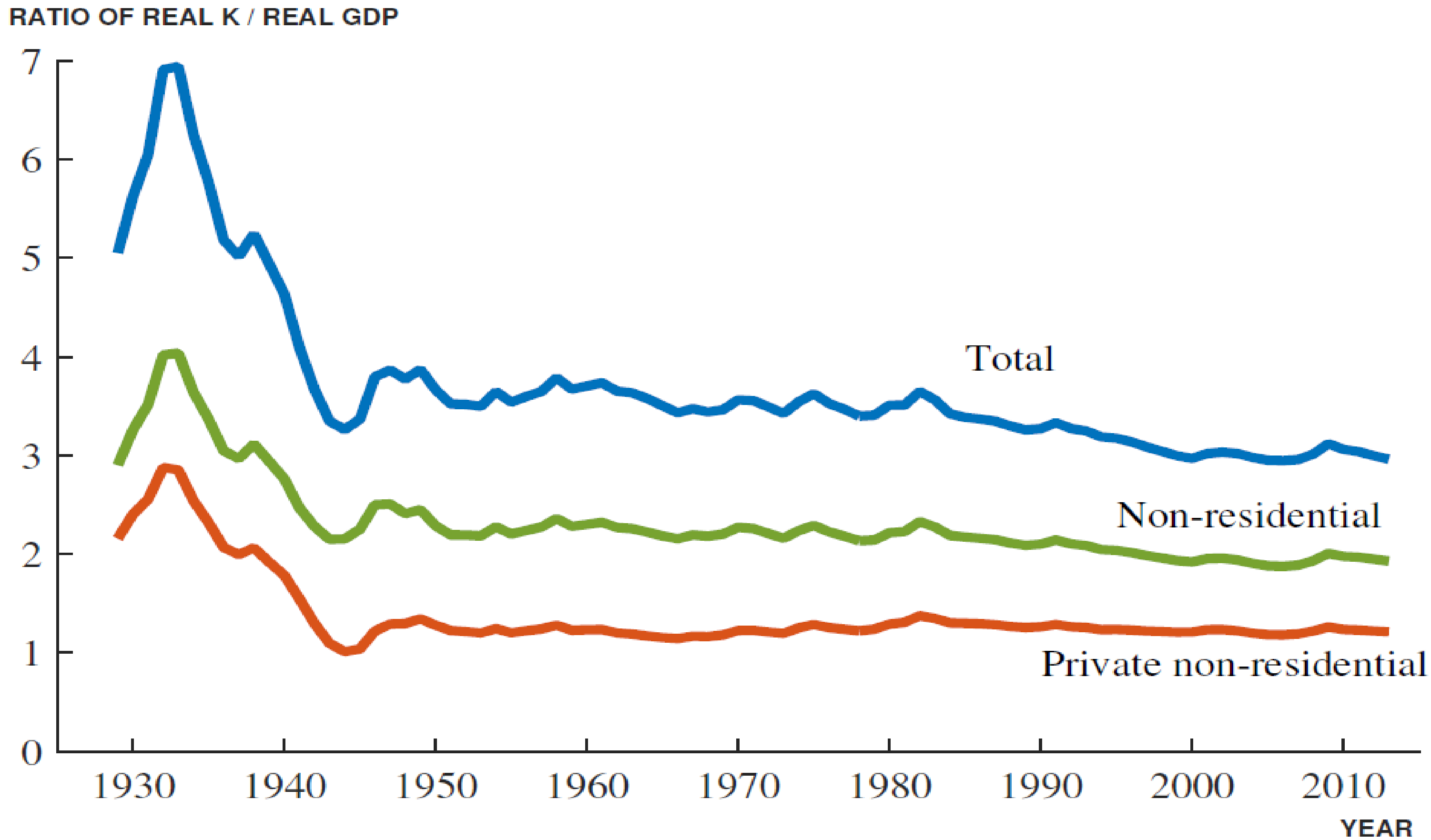
- **Physical/Human Capital Accumulation and Growth**
  - **I/Y: investment rate**
  - **H: human capital index (year of schooling or PWT index)**

Country	I/Y (%)	Y_i/Y_{US}*100 (1990)	$\triangle Y_i/Y_i$ (%)		Country	H Index	Y_i/Y_{US}*100 (1990)	$\triangle Y_i/Y_i$ (%)
U.S.	24.0	100	2.1		U.S.	11.8	100	2.1
Algeria	23.3	14	2.2		Argentina	6.7	19	0.7
Zambia	27.9	4	-0.8		Philippines	6.7	14	1.3
Guyana	25.1	7	-0.9		Korea	9.2	45	6.3
Japan	36.6	80	5.6		New Zealand	12.3	63	1.4
Singapore	32.6	60	6.4		Norway	10.6	81	3.7

- **Physical capital:**

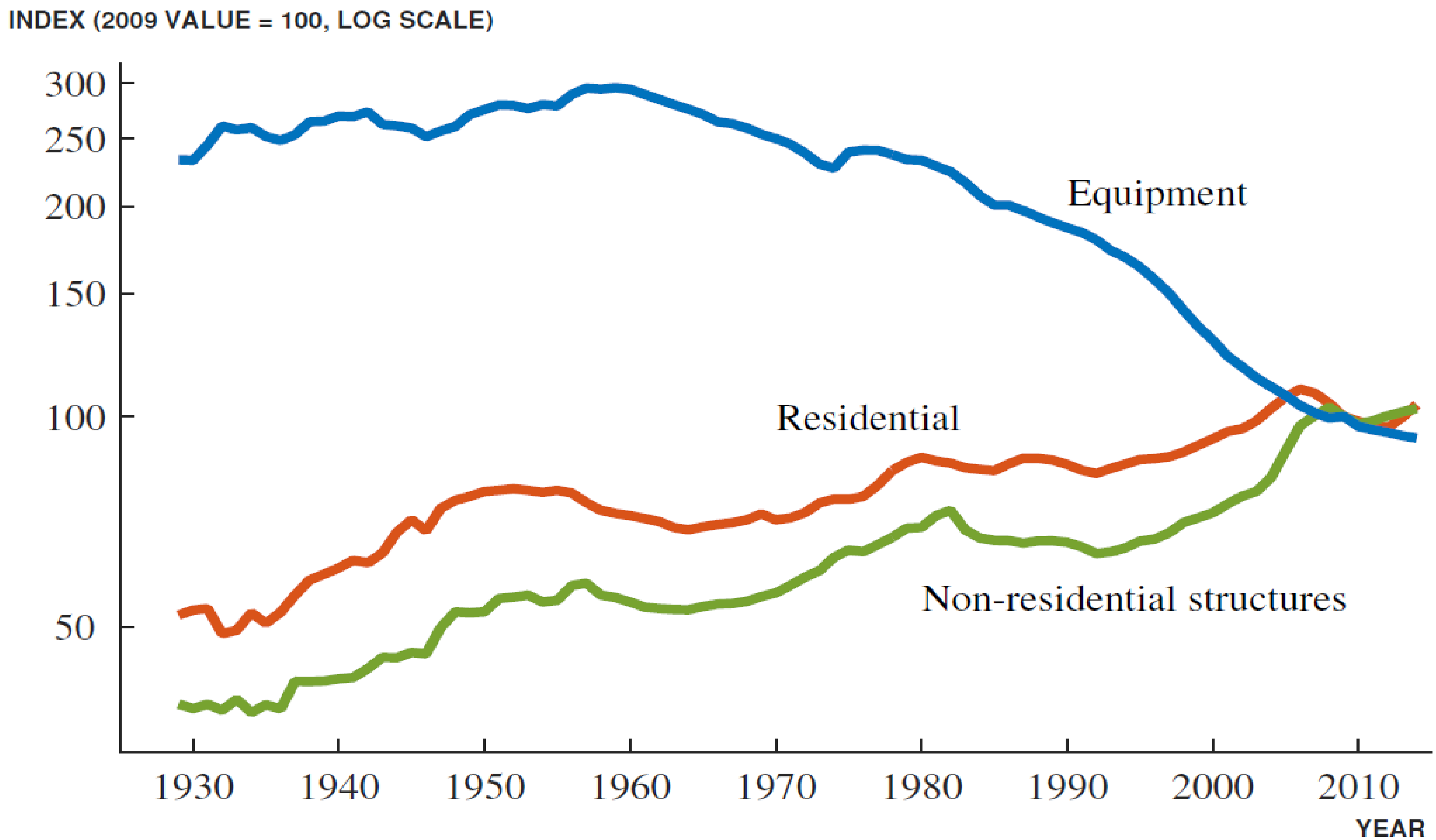


U.S. Capital-Output Ratio



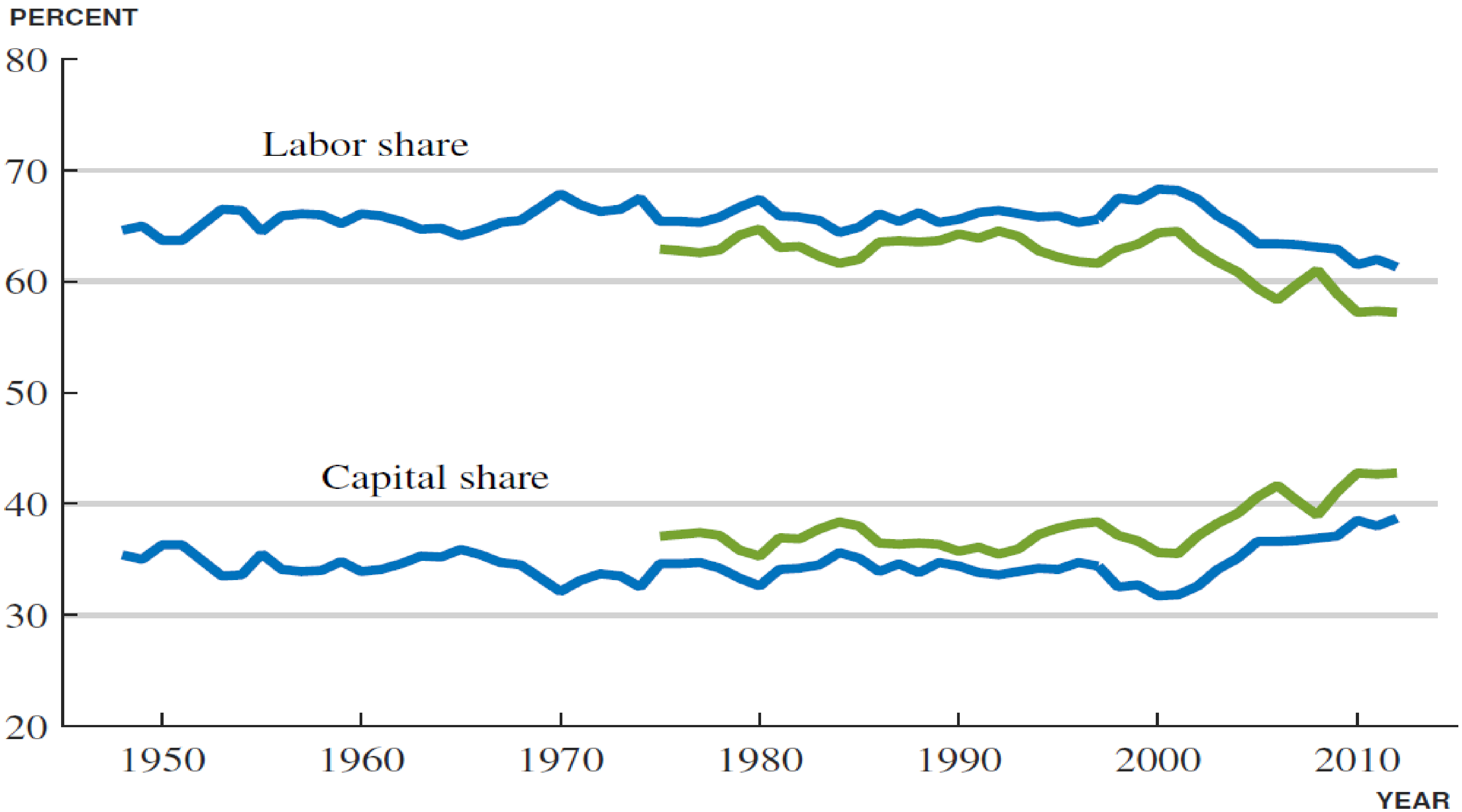
- Destruction of physical capital by wars
- Rising service sector

U.S. Relative Price of Investment



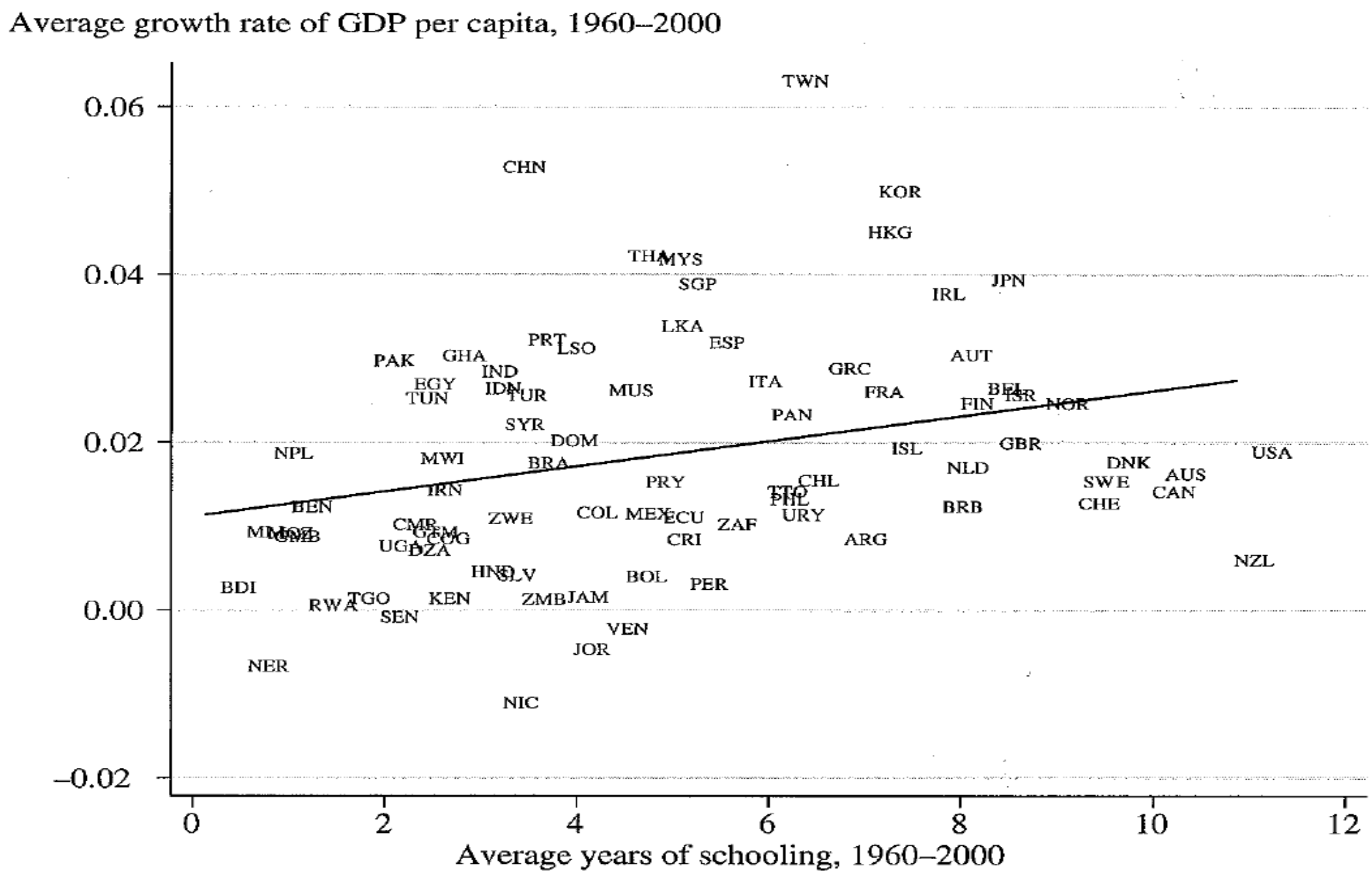
- **Fallen equipment price due to computerization and mass production**
- **Rising commercial structure and housing prices (land)**

U.S. Capital Shares

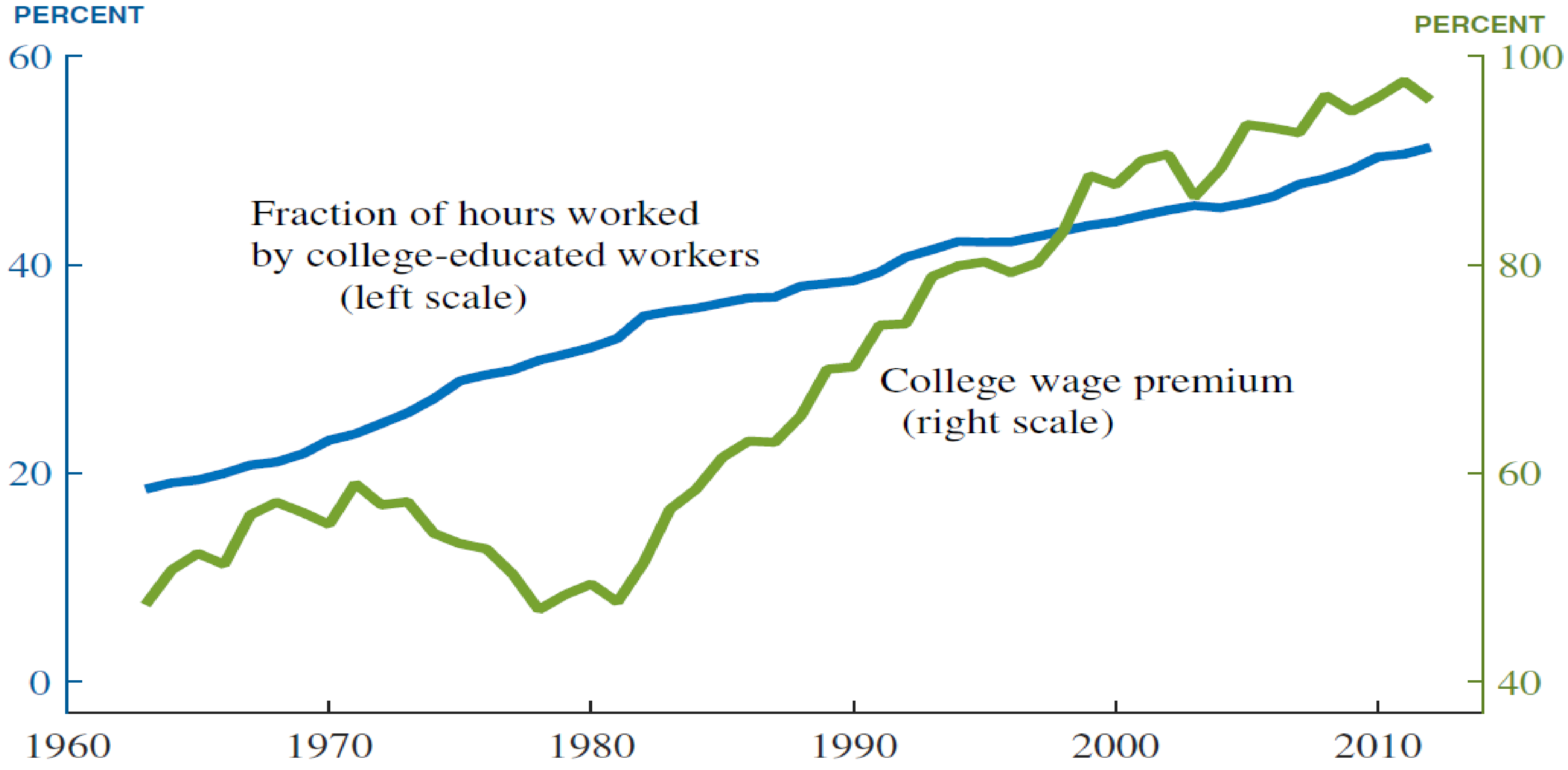


- **Declined labor share:**
  - **role played by automation**
  - **implication for rising inequality, especially top inequality**

- **Human capital:**

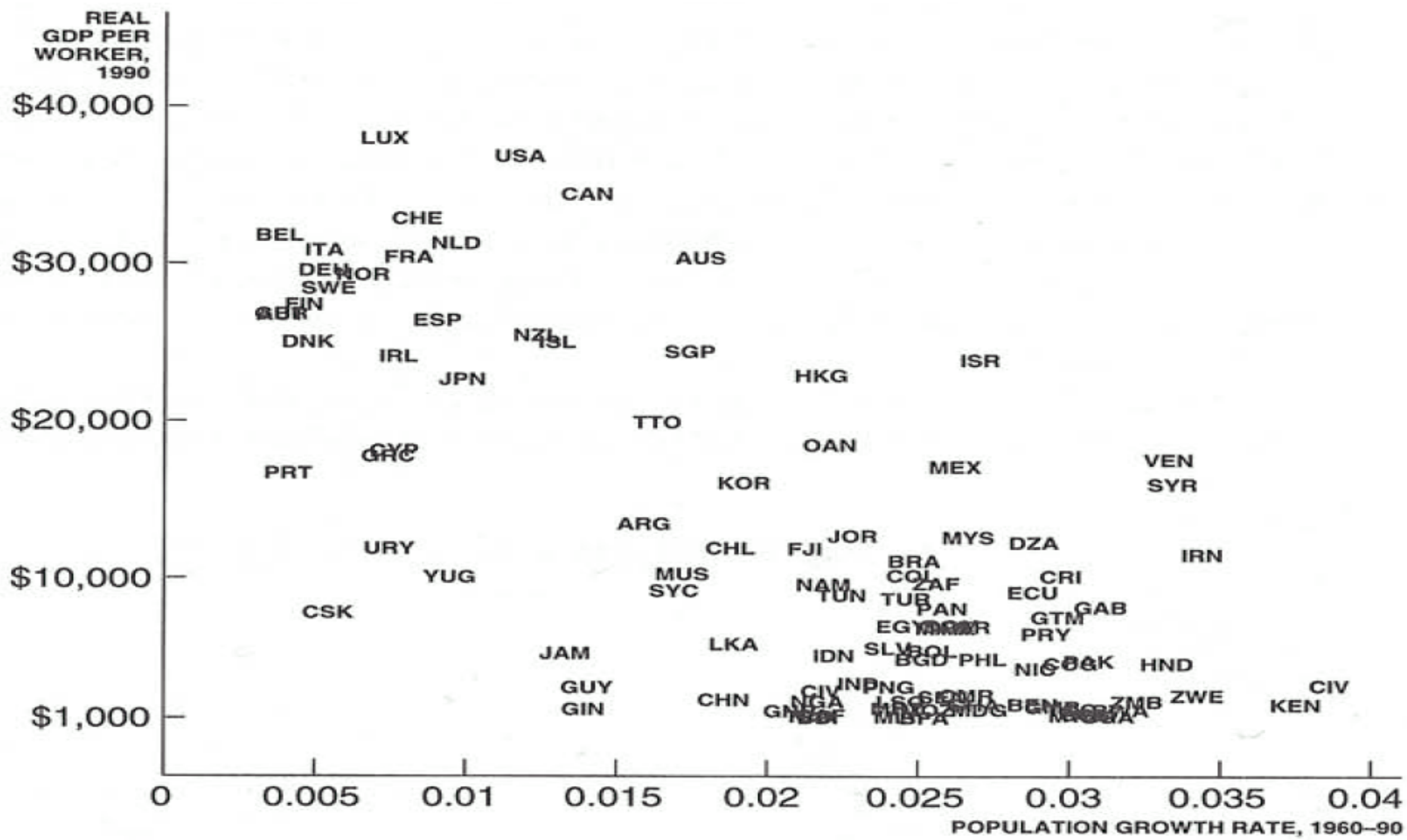


U.S. Skilled Labor Growth and Skill Premium



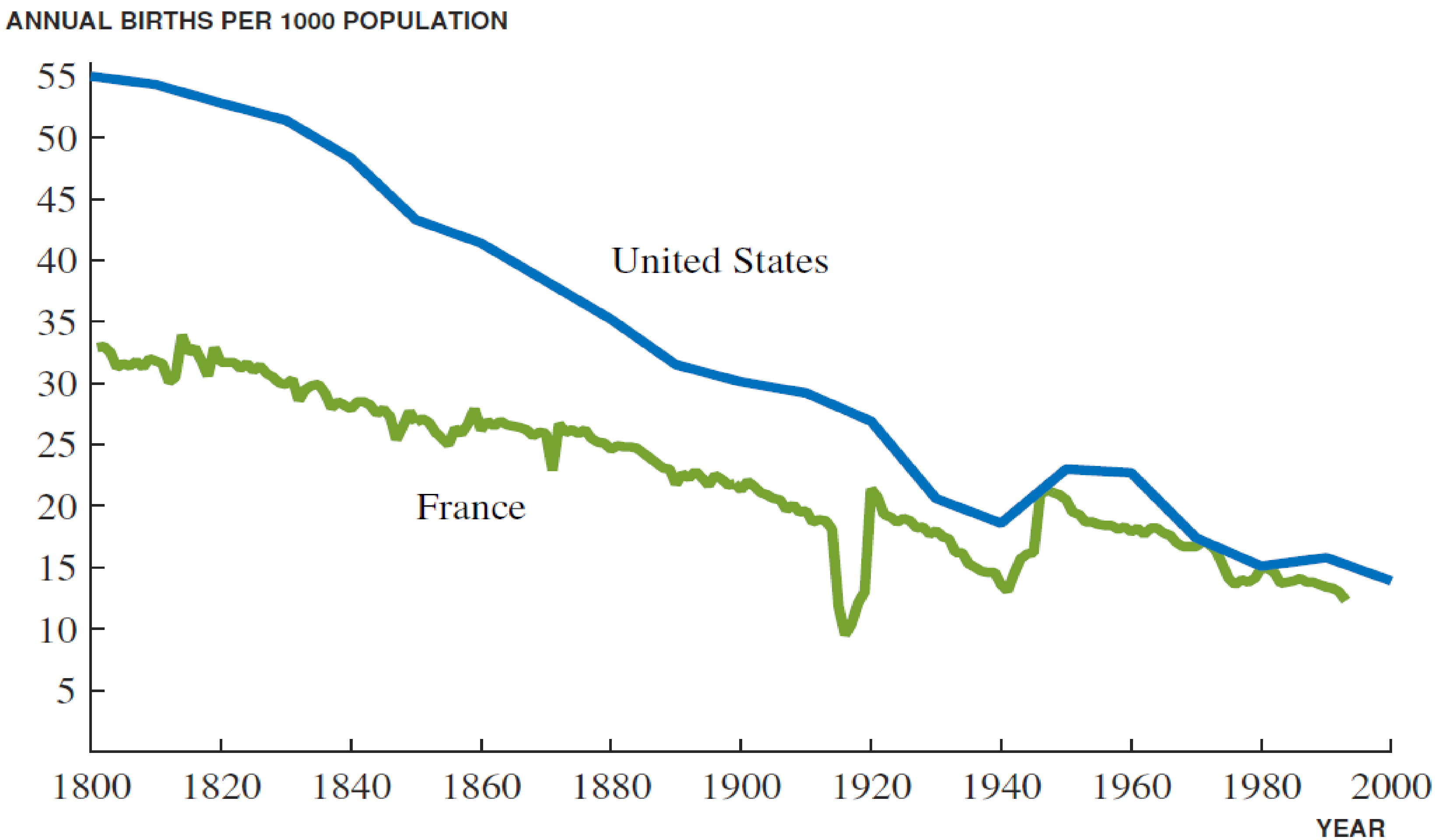
- Skilled: 14 or 16 years of schooling (developing or developed)
- Rising skill premium (relative wage) since 1980

- **Population growth:**

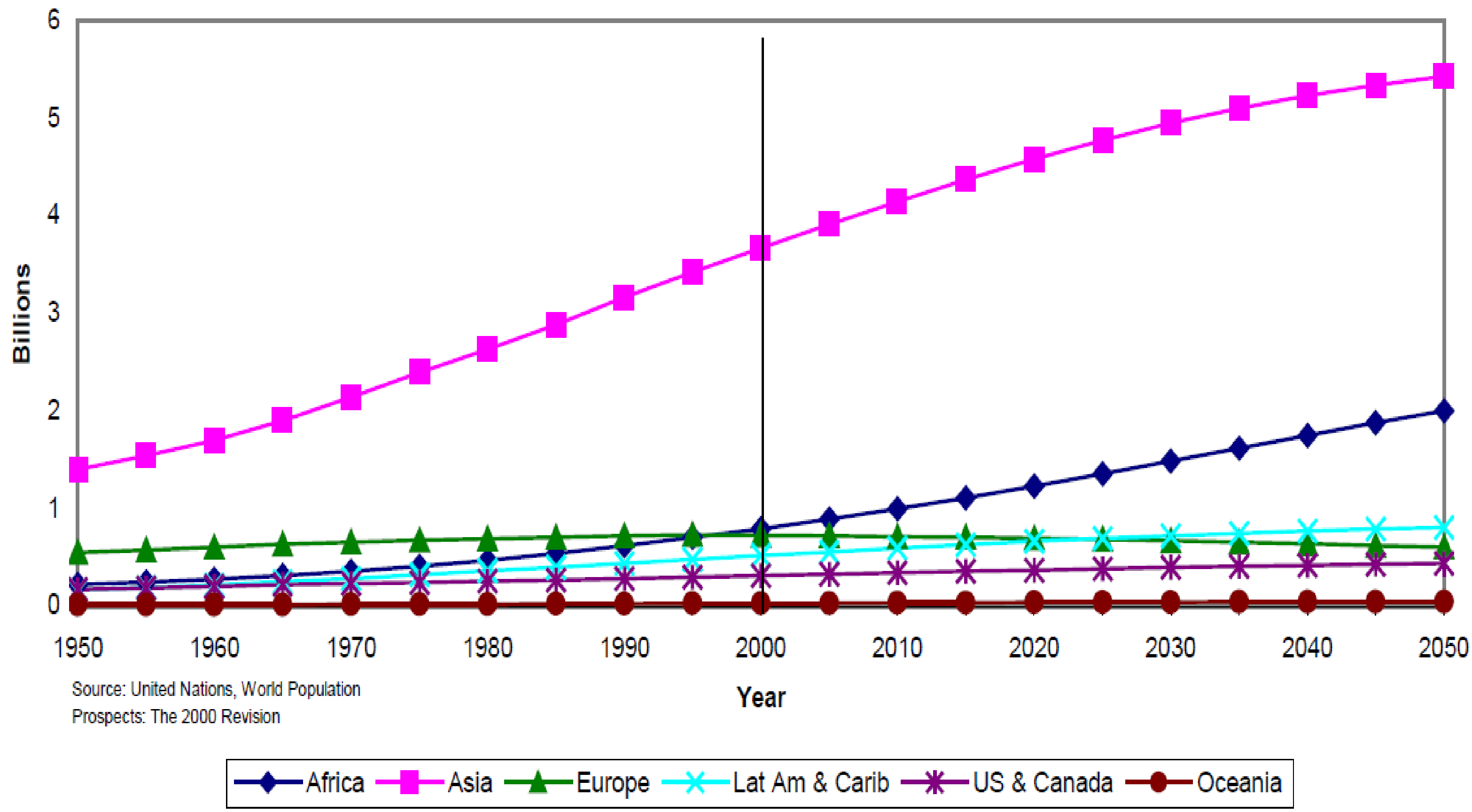


- Negative relationship: cake eating
- Quantity-quality tradeoff in fertility choice (Becker)

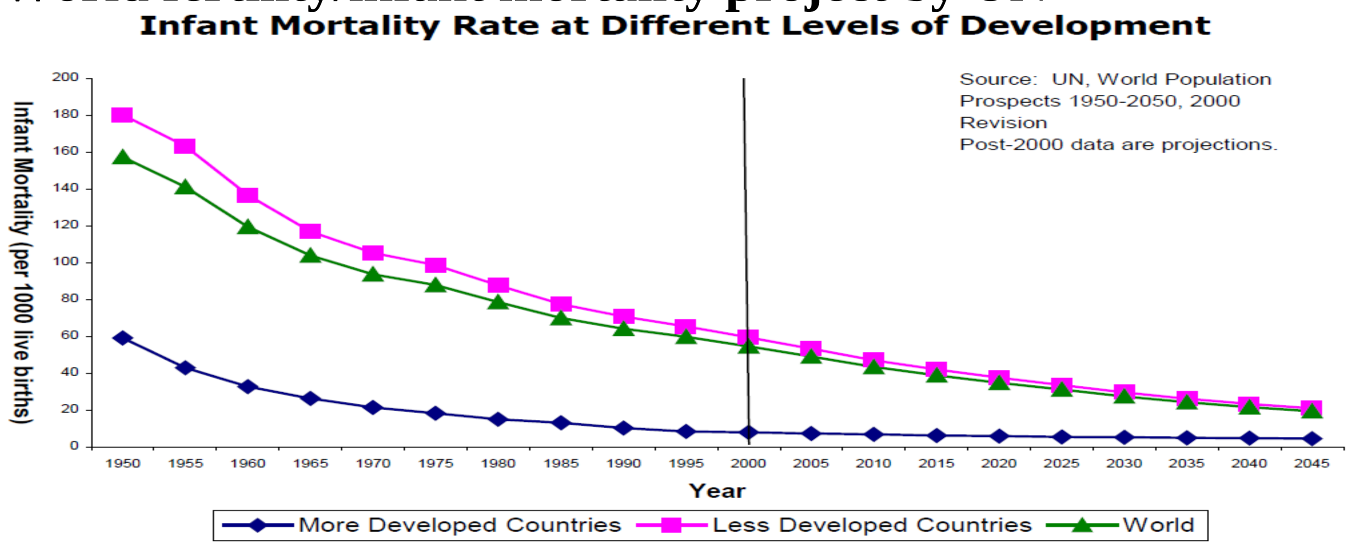
o Fertility of Advanced Countries: US vs. France

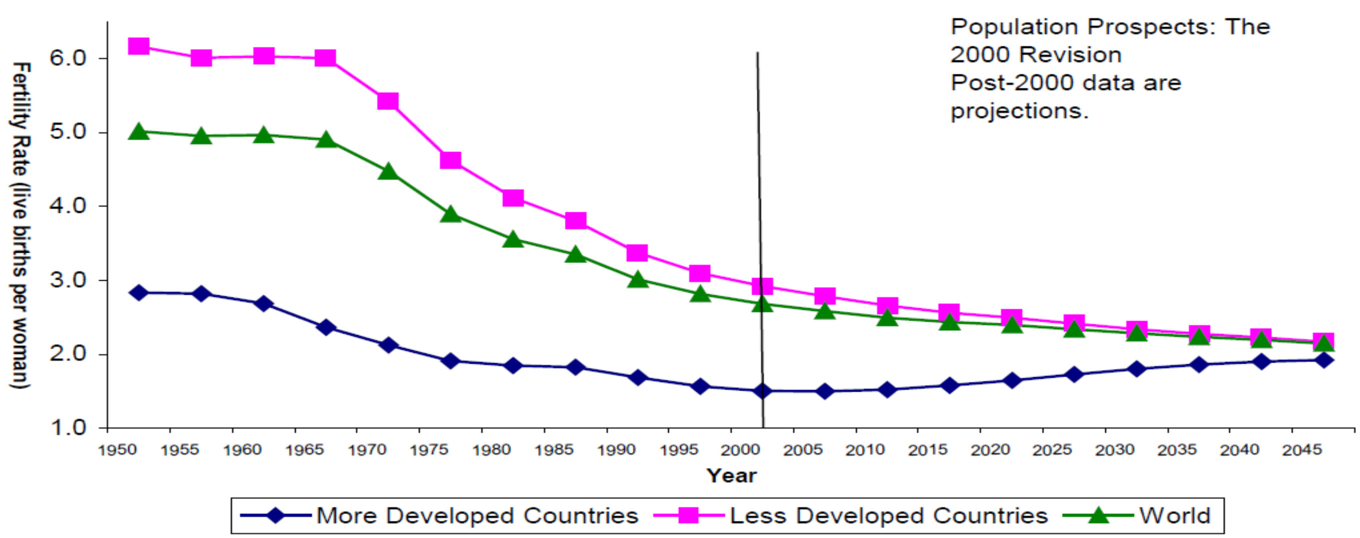


- **France: socialism/high welfare toward poor & children => more moderate decline in total fertility** o **World population projection by UN (Bloom-Canning-Sevilla)**

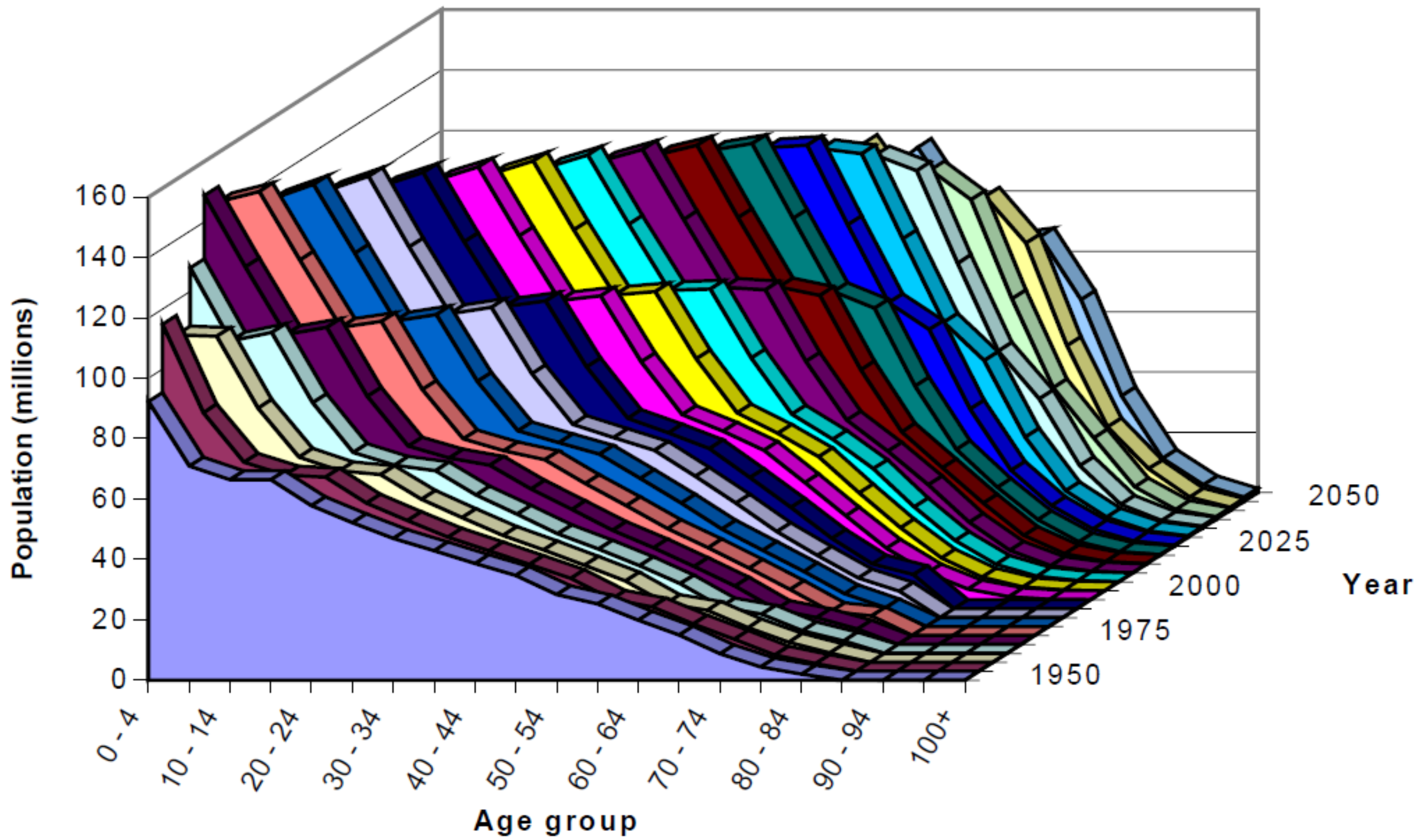


- **World fertility/infant mortality project by UN**

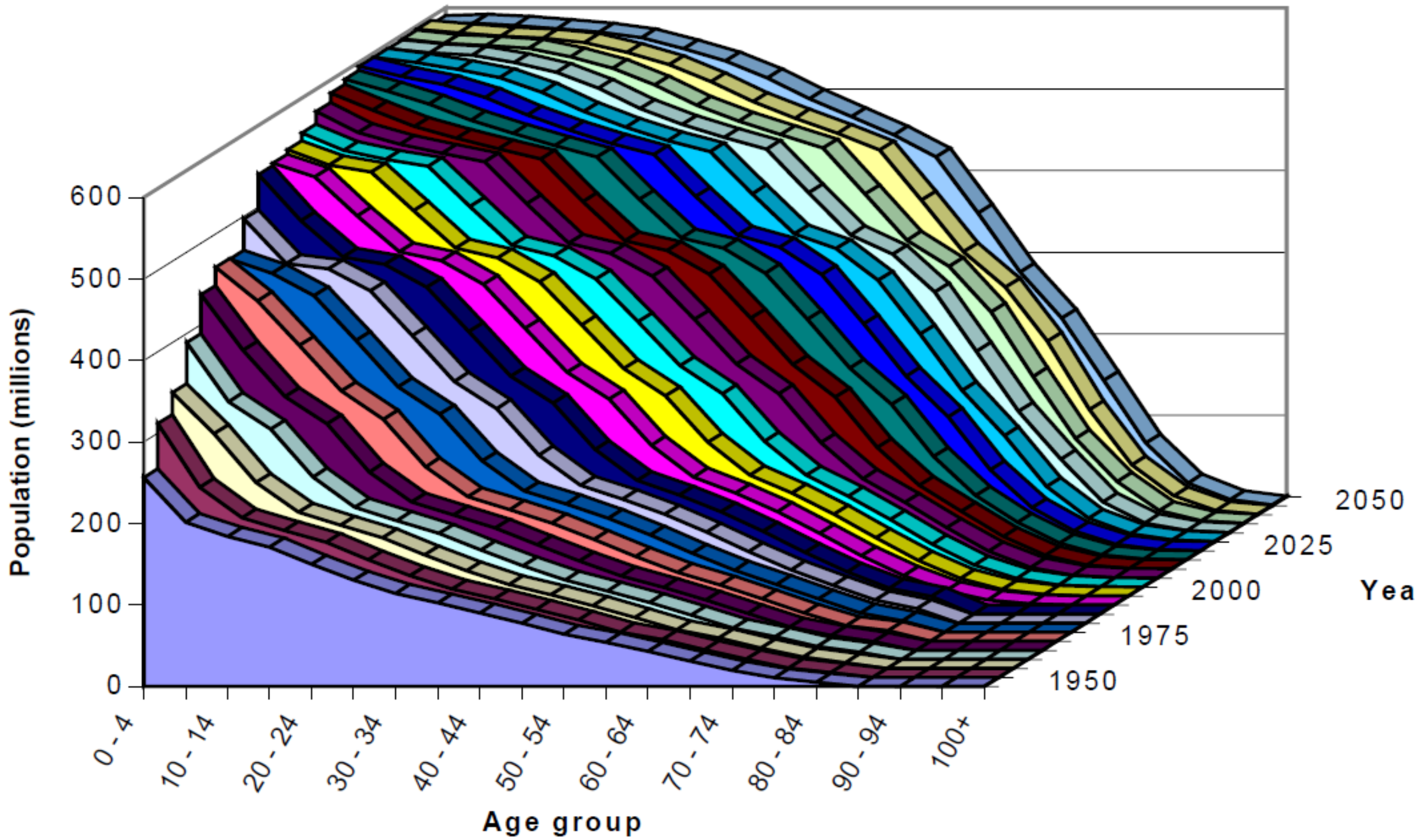




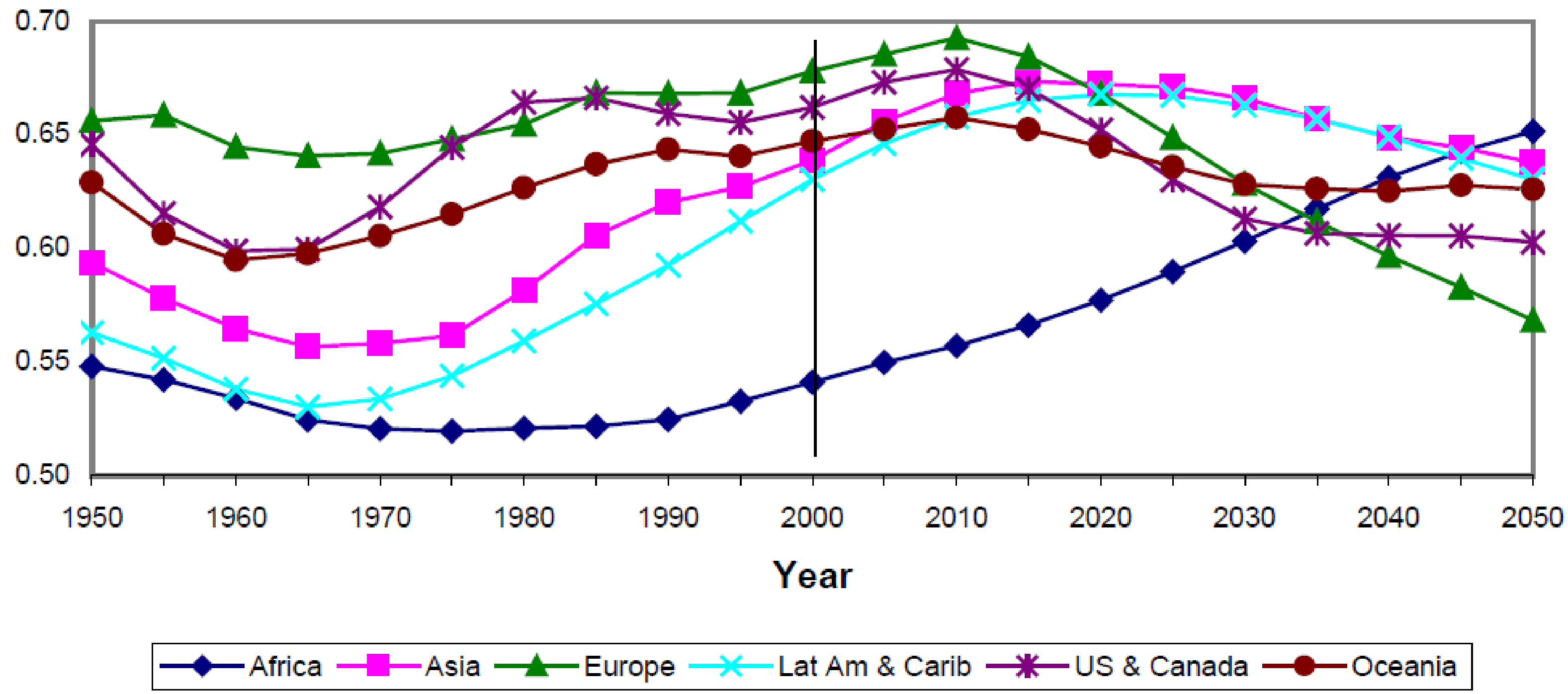
- **Population ageing:**
  - **East Asia**



- **LDCs**

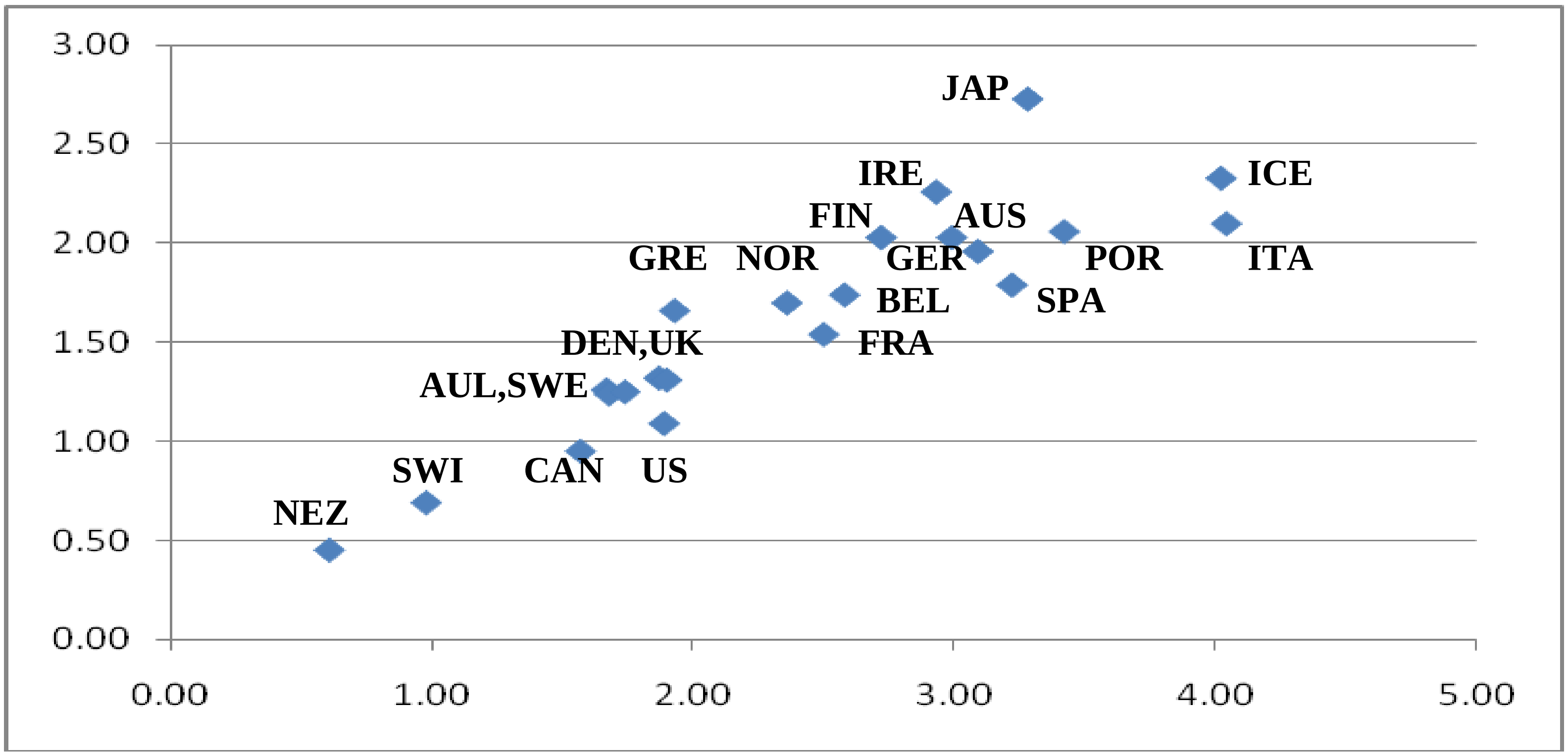

Population (millions)
600
500
400
300
200
100
0
0 - 4
10 - 14
20 - 24
30 - 34
40 - 44
50 - 54
60 - 64
70 - 74
80 - 84
90 - 94
100+
Age group
1950
1975
2000
2025
2050
Year

o **Share of working-age population:**



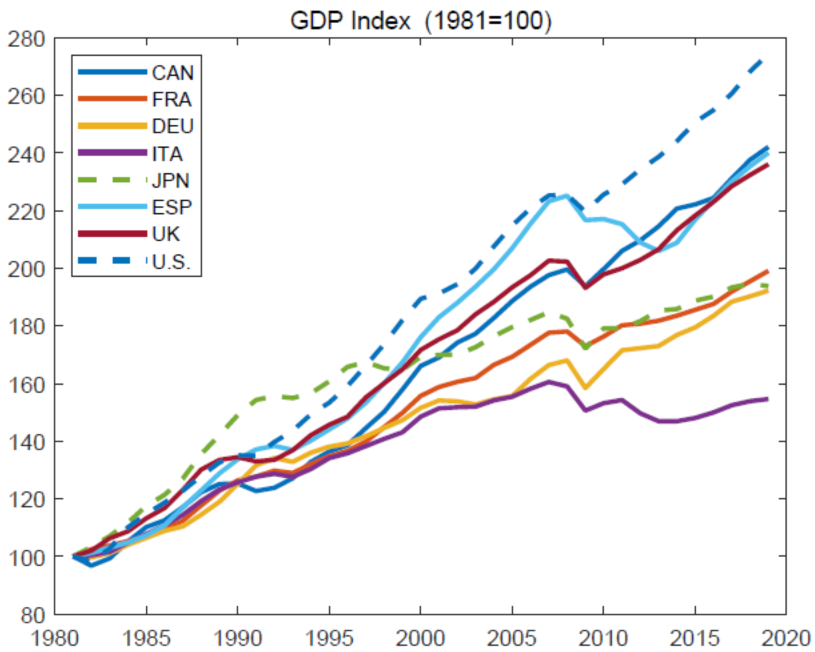
- **TFP growth: PWT TFP index**

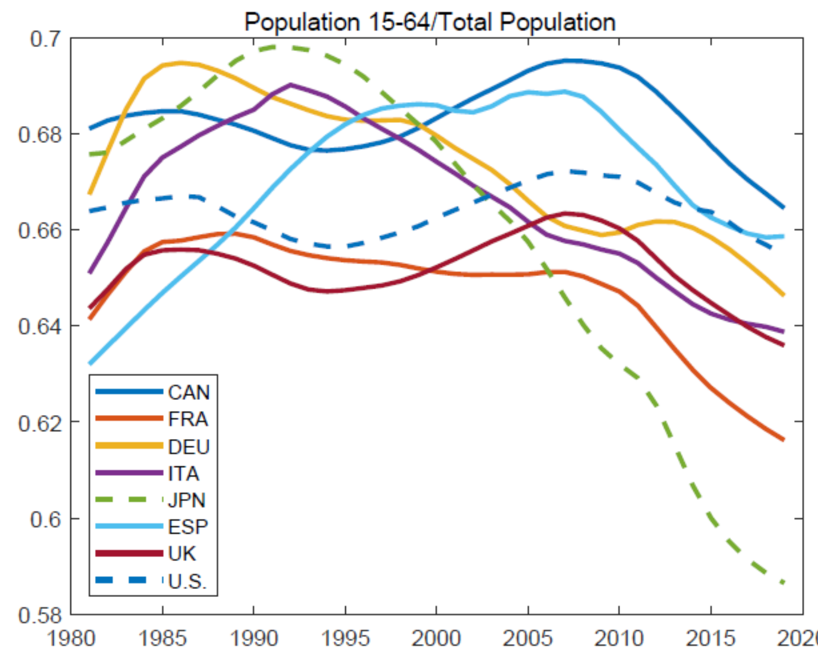
TFP growth versus per capita real GDP growth



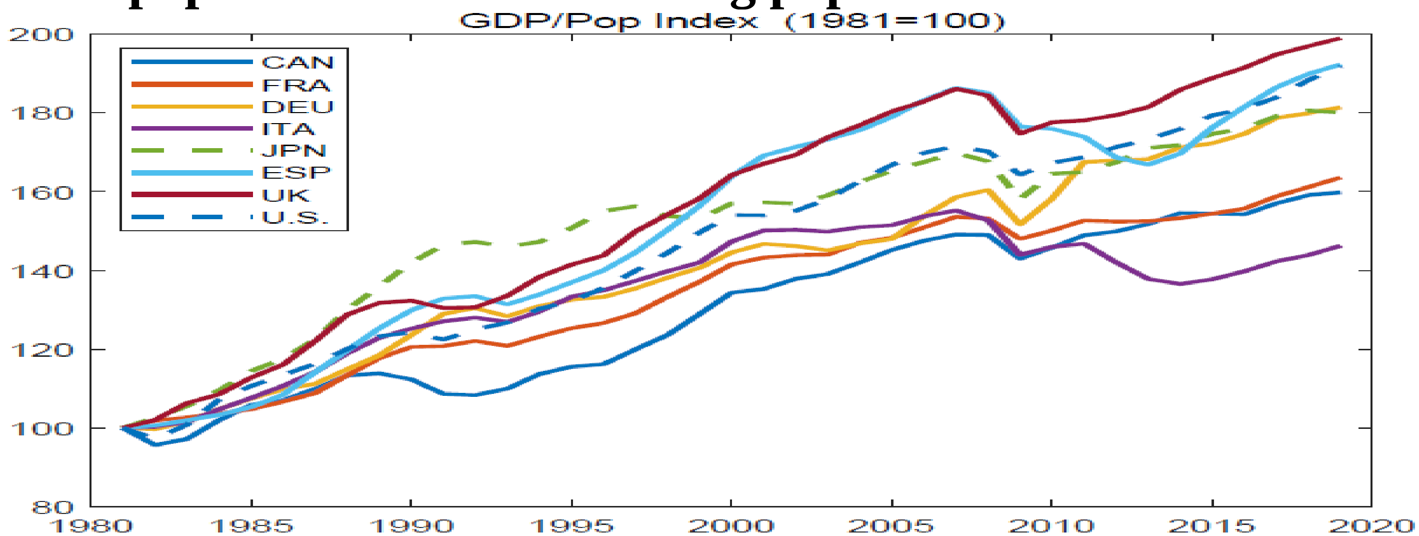
per capita real GDP growth

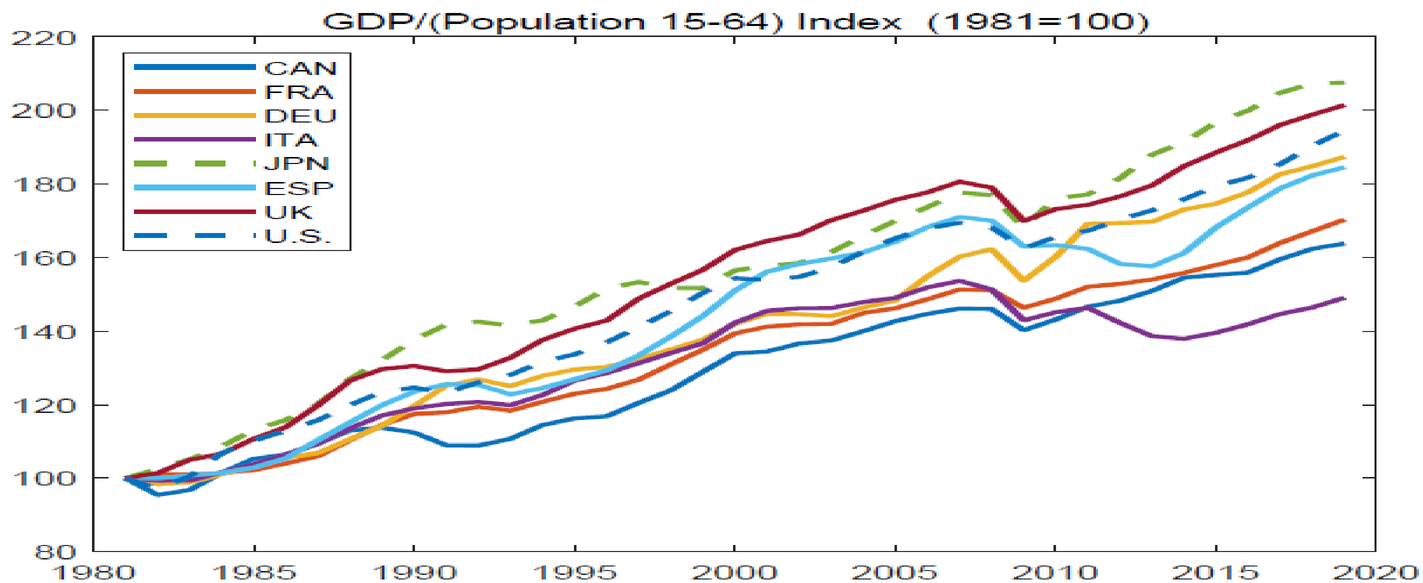
- **GDP index and working pop (15-64) share: G7 + Spain (Fernandes Villaverde-Ventura-Yao 2023)**





- **GDP/pop index vs. GDP/working pop index**





- **GDP per capita measure can lead to sizable bias, especially since the Great Recession**

Table 4: G7 plus Spain: Basic Growth and Population Facts, 1981-2007

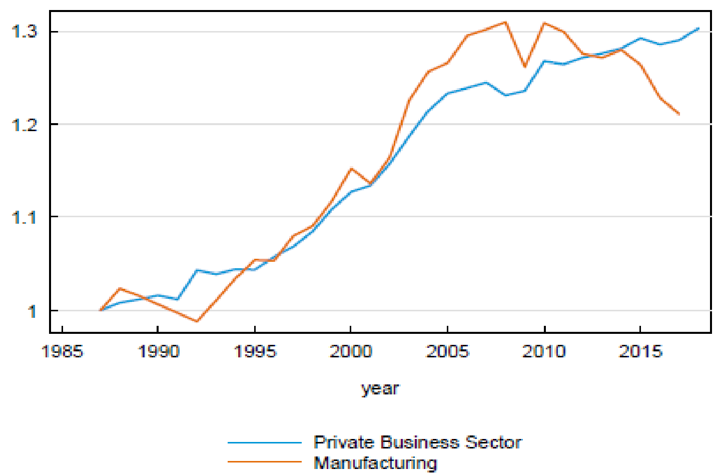
1981-2007	Canada	France	Germany	Italy	Japan	Spain	UK	USA
GDP	2.68	2.24	1.99	1.84	2.41	3.15	2.76	3.19
GDP per Capita	1.57	1.67	1.80	1.71	2.08	2.44	2.43	2.11
Population	1.09	0.56	0.19	0.13	0.32	0.70	0.33	1.05
GDP per Working-age Adult	1.49	1.61	1.84	1.67	2.25	2.10	2.31	2.06
Working-age Population	1.17	0.62	0.15	0.17	0.15	1.03	0.44	1.10
Working-age Pop. Ratio	0.68	0.65	0.68	0.67	0.68	0.67	0.65	0.66

Table 5: G7 plus Spain: Basic Growth and Population Facts, 2008-2019

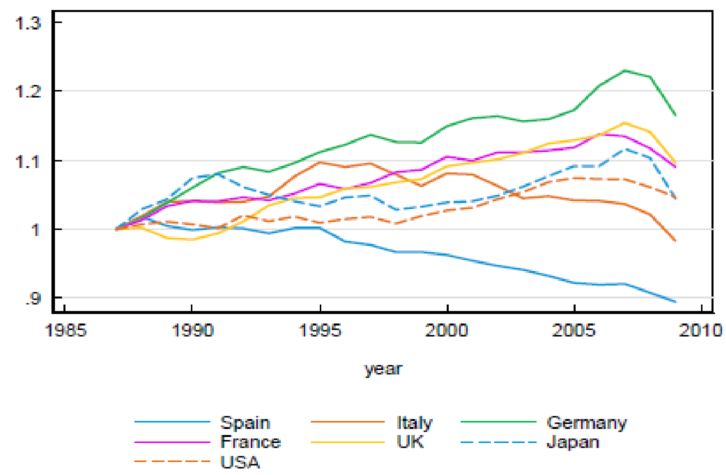
2008-2019	Canada	France	Germany	Italy	Japan	Spain	UK	USA
GDP	1.79	1.03	1.27	-0.23	0.58	0.61	1.43	1.81
GDP per Capita	0.65	0.61	1.16	-0.36	0.68	0.38	0.71	1.11
Population	1.13	0.42	0.11	0.14	-0.10	0.23	0.71	0.70
GDP per Working-age Adult	1.07	1.11	1.35	-0.11	1.49	0.78	1.10	1.34
Working-age Population	0.71	-0.07	-0.08	-0.12	-0.90	-0.16	0.33	0.46
Working-age Pop. Ratio	0.68	0.63	0.66	0.65	0.61	0.67	0.65	0.66

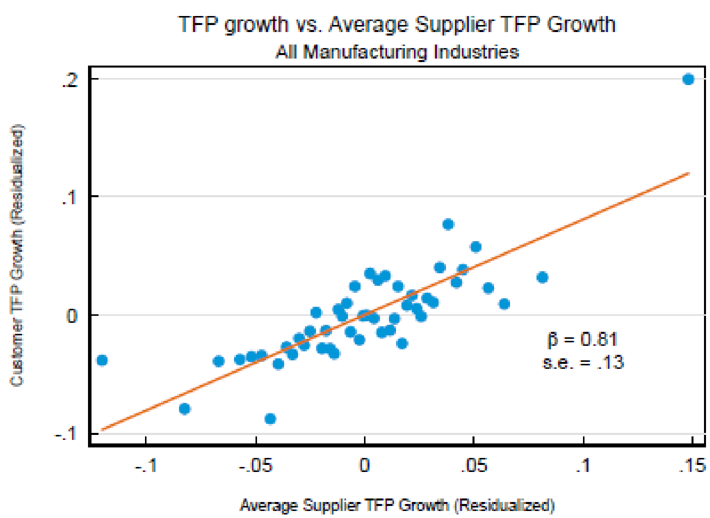
- **Sectoral composition effect: Acemoglu-Autor-Patterson (2023)**

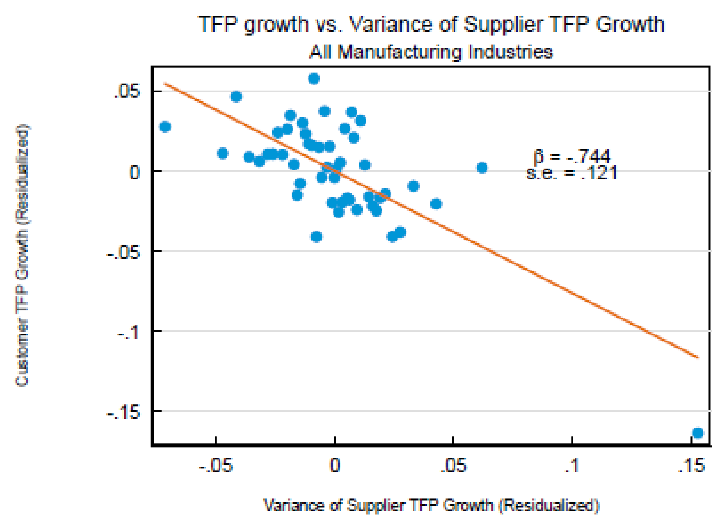
(a) United States



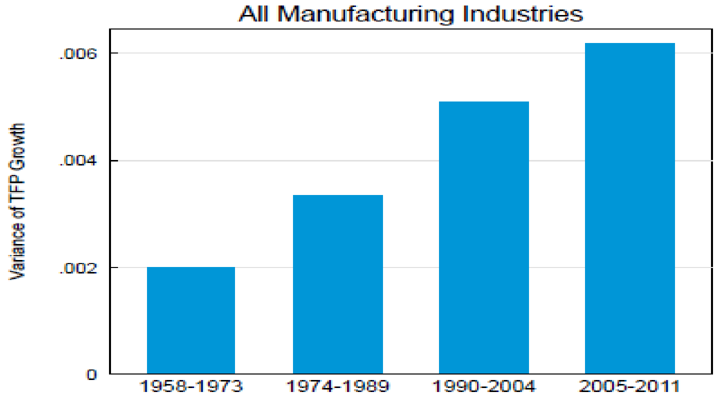
(b) Select OECD Countries

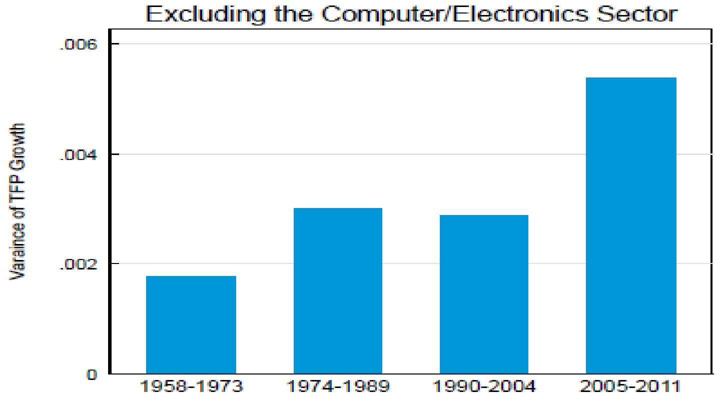






o Upstream suppliers matter





	Downstream		Upstream Average		Upstream Variance	
	Mean	SD	Mean	SD	Mean	SD
Panel A: Manufacturing Industries						
Growth in log(TFP)	.018	.152	.033	.075	.022	.048
Growth in log(Patents)	.132	.19	.085	.119	.015	.012
Growth in Price Index	.134	.178	.125	.172	.033	.059
Growth in log(Employment)	-.08	.258	-.087	.115	.027	.021
Panel B: All Industries						
Growth in log(TFP)	.015	.155	.034	.079	.028	.057
Growth in Price Index	.095	.147	.081	.145	.04	.069
Growth in log(Employment)	-.079	.266	-.084	.123	.025	.023
Panel C: International panel						
Growth in log(TFP)	.046	0.16	.041	.068	.018	.023

o The ups and downs of industries

Panel A: List of Fastest-Growing Industries that Drive Rising TFP Variance

1997–2002 Industries	*2002–2007 Industries*
Semiconductor and Related Devices	Semiconductor and Related Devices
Electronic Computers	Electronic Computers
Paper (except Newsprint) Mills	Computer Storage Devices
Other Animal Foods	Sawmills
Iron and Steel Mills	Biological Products (except Diagnostic)
All Other Plastics Products	Other Basic Inorganic Chemicals
Motor Vehicle Electrical and Electronic Equipment	Other Plastics Products
Soybean Processing	Motor vehicle transmission and power train parts
Gas engine and engine parts	Motor vehicle metal stamping
Motor Vehicle Metal Stamping	Petrochemicals

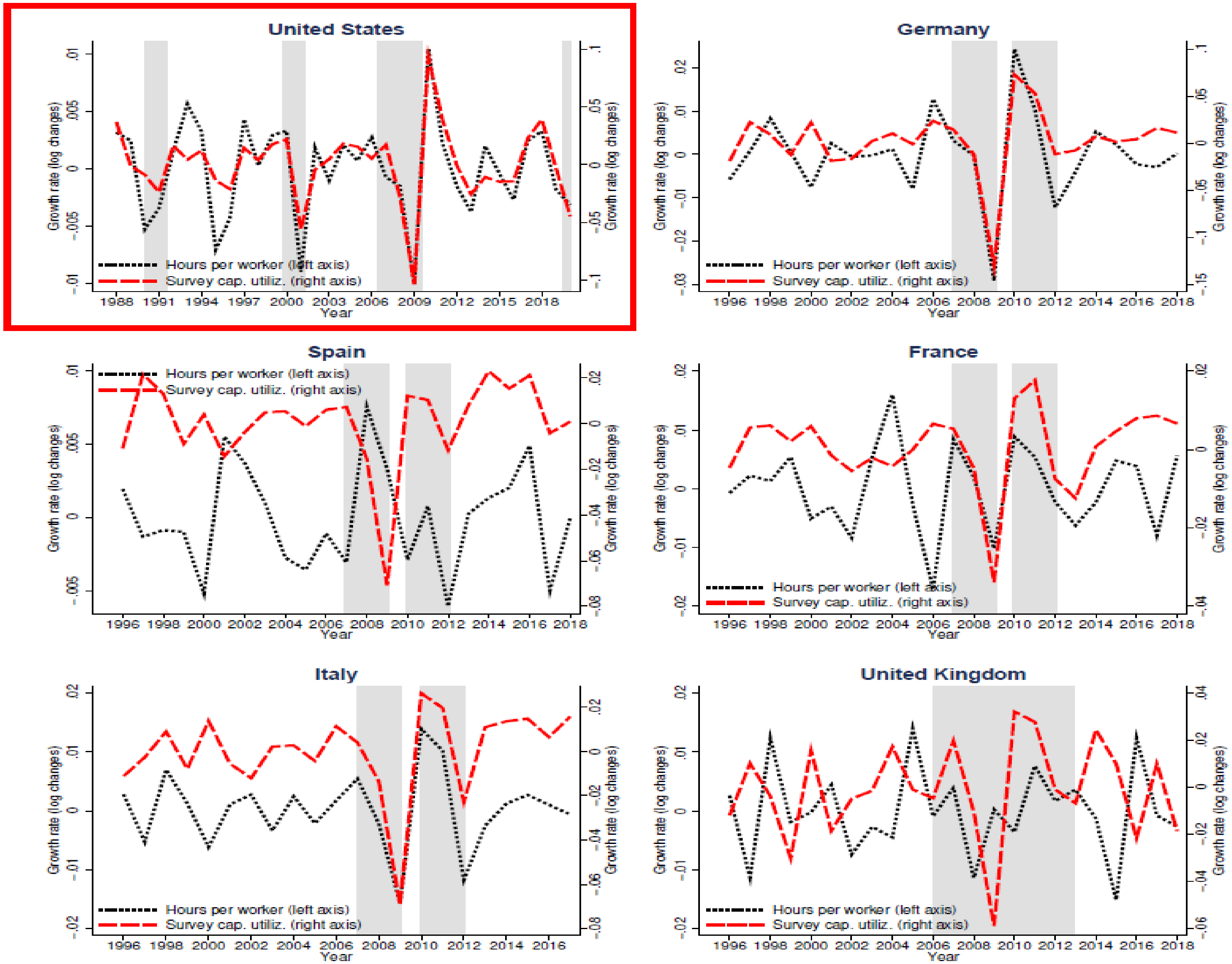
Panel B: List of Bottleneck Industries

1997–2002 Industries	*2002–2007 Industries*
Commercial Lithographic Printing	Petroleum Refineries
All Other Basic Organic Chemical	Pharmaceutical Preparation
Printed Circuit Assembly (Electronic Assembly)	Other Communication and Energy Wires
Corrugated and Solid Fiber Boxes	Manifold Business Forms Printing
Petrochemicals	Corrugated and Solid Fiber Boxes
Radio/TV Broadcasting	Rolled Steel Shape Manufacturing
Bare Printed Circuit Boards	Turbine and Turbine Generator Set Units
Electronic Connectors	Medicinal and Botanical Manufacturing
Other Electronic Components	Motor Vehicle Electrical and Electronic Equipment
Electronic Capacitors	Unsupported Plastics Film and Sheets

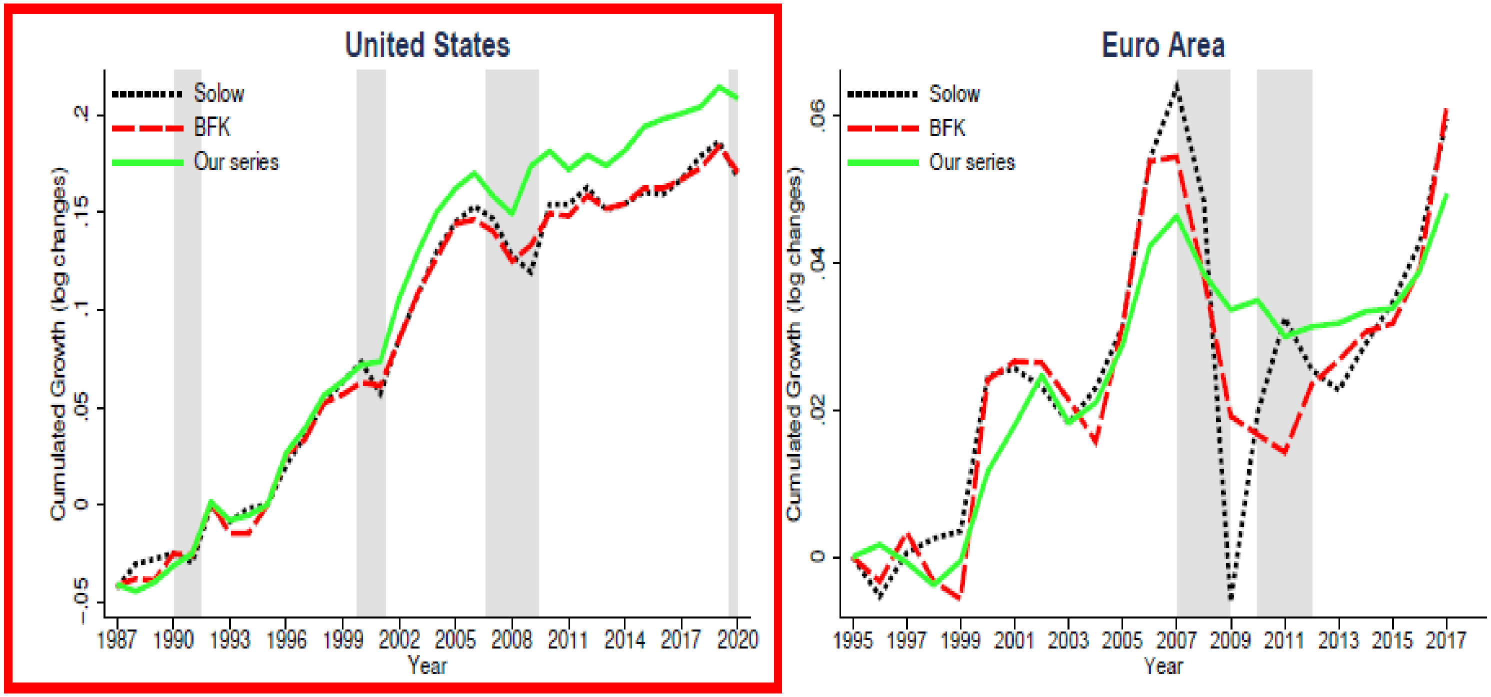
- **Factor utilization and TFP: Comin-Quintana-Schmitz-Trigari (2023)**
  - **Average output elasticities**

	USA	Germany	Spain	France	Italy	UK
Materials						
Our elasticity	0.43	0.54	0.55	0.56	0.59	0.53
Solow-BFK elasticity	0.41	0.52	0.52	0.53	0.56	0.50
Labour						
Our elasticity	0.41	0.34	0.33	0.35	0.31	0.37
Solow-BFK elasticity	0.39	0.33	0.32	0.34	0.29	0.35
Capital						
Our elasticity	0.17	0.12	0.12	0.09	0.10	0.09
Solow-BFK elasticity	0.20	0.14	0.16	0.13	0.15	0.15

o **labor utilization (hours worked) and capital utilization**



o **cumulated TFP growth**



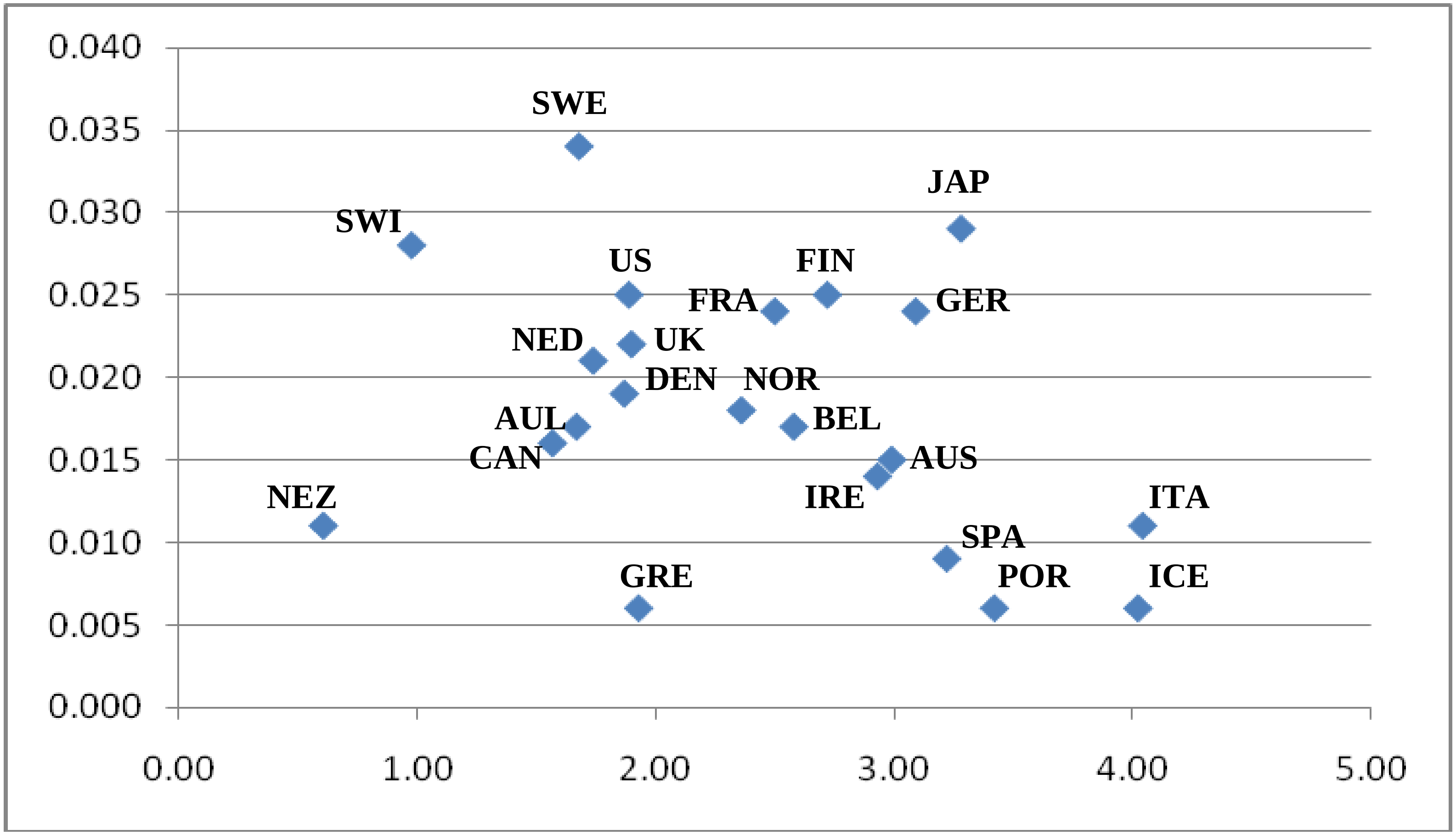
Notes: BFK = Basu-Fernald-Kimball (AER 2006)

- **average TFP growth rates**

	USA	EA	Germany	Spain	France	Italy	UK
Overall sample							
Solow residual	0.64	0.27	0.73	-0.33	0.28	-0.30	0.91
BFK method	0.64	0.28	0.76	-0.33	0.26	-0.33	0.92
Our method	0.76	0.22	0.61	-0.40	0.25	-0.27	1.11
Subperiods, our method							
1988-2004	1.13	.	.	.	.	.	.
2004-2009	0.47	.	.	.	.	.	.
2009-2020	0.32	.	.	.	.	.	.
1995-2007	.	0.39	0.82	-0.72	0.88	-0.26	1.73
2008-2018	.	0.03	0.38	-0.06	-0.43	-0.28	0.44

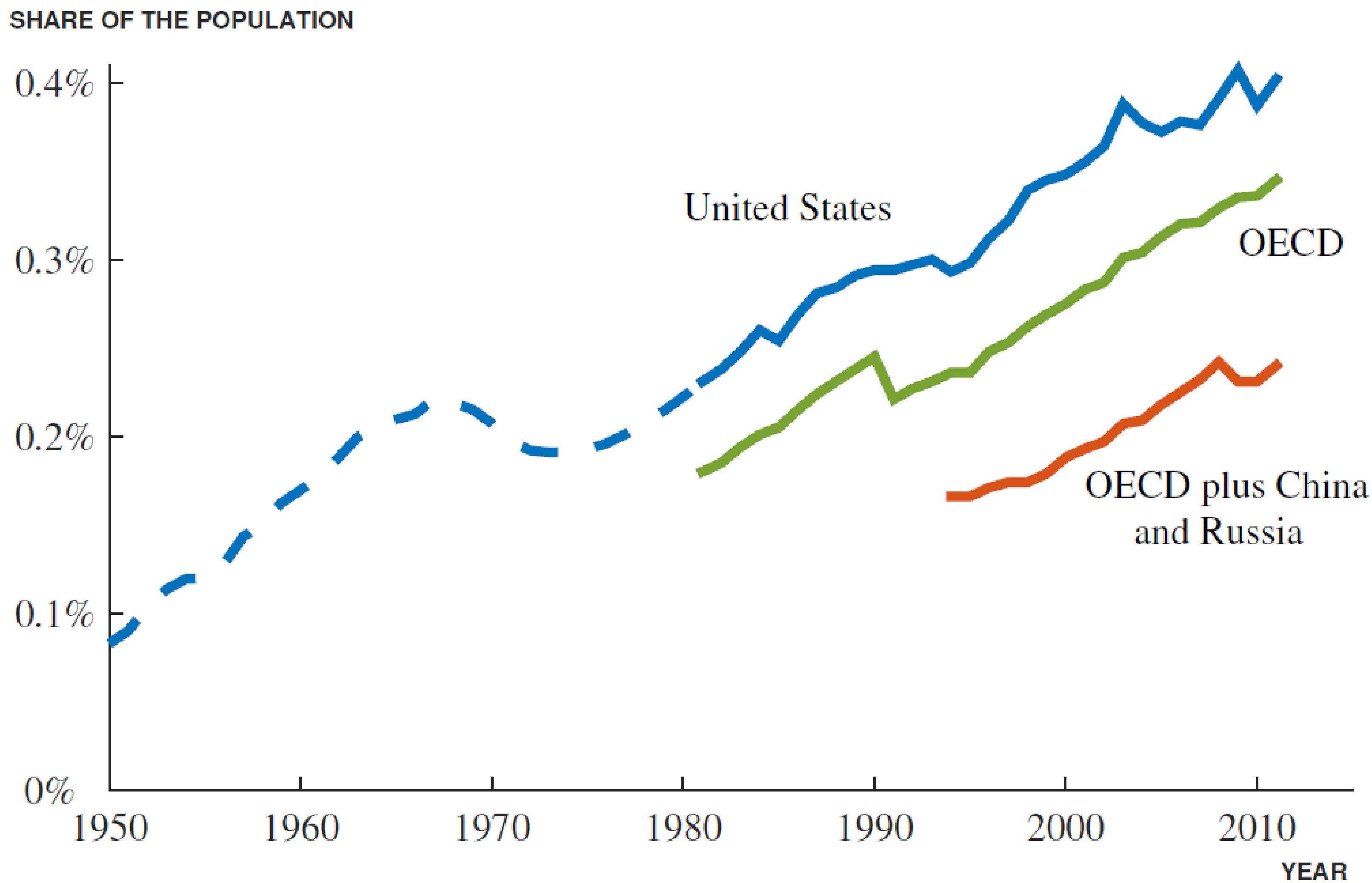
- **R&D: other forms of technical progress (licensing, imitation, technology spillovers, technology assimilation)**

R&D-GDP Share

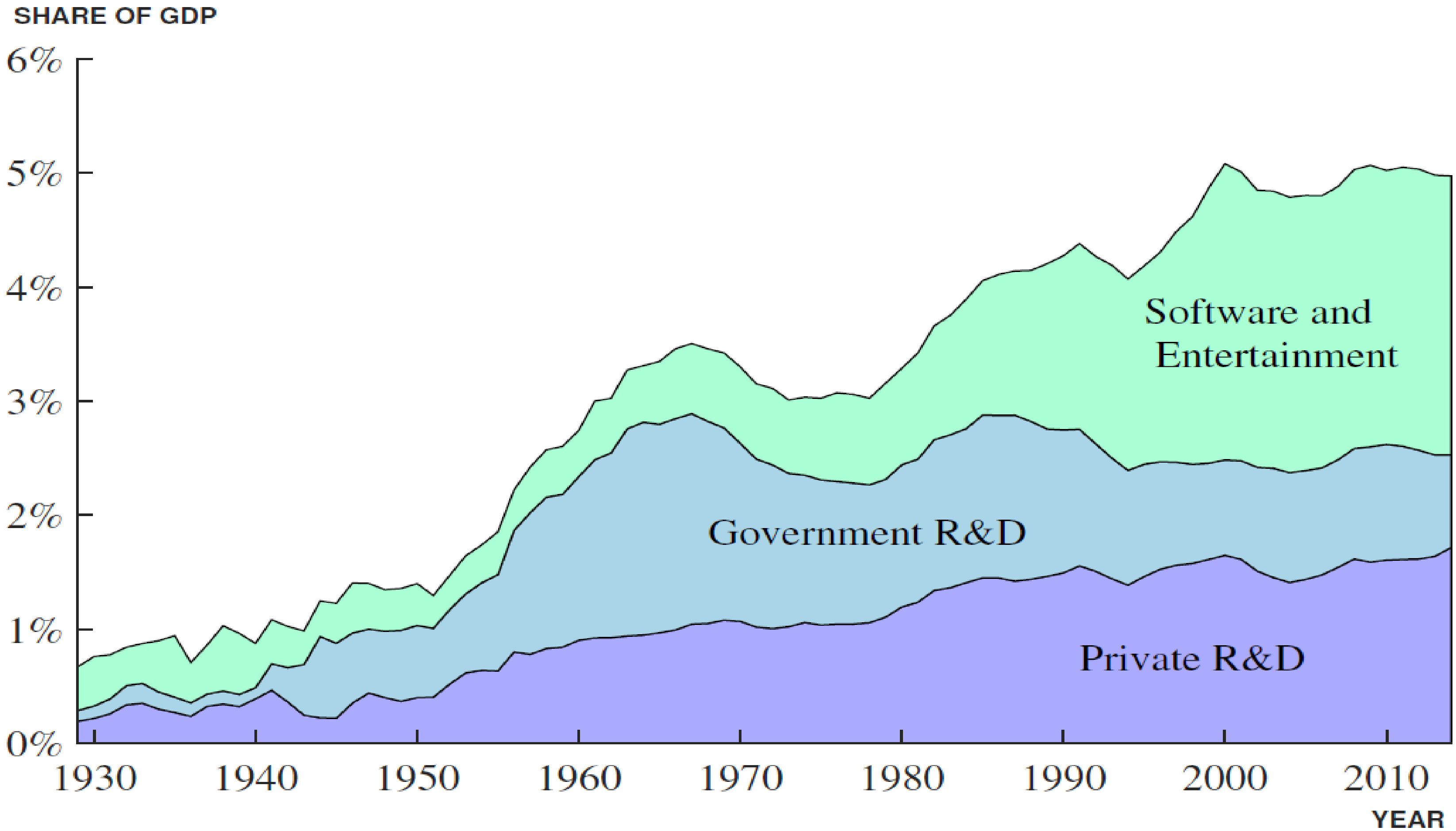


per capita real GDP growth

R&D Employment Share

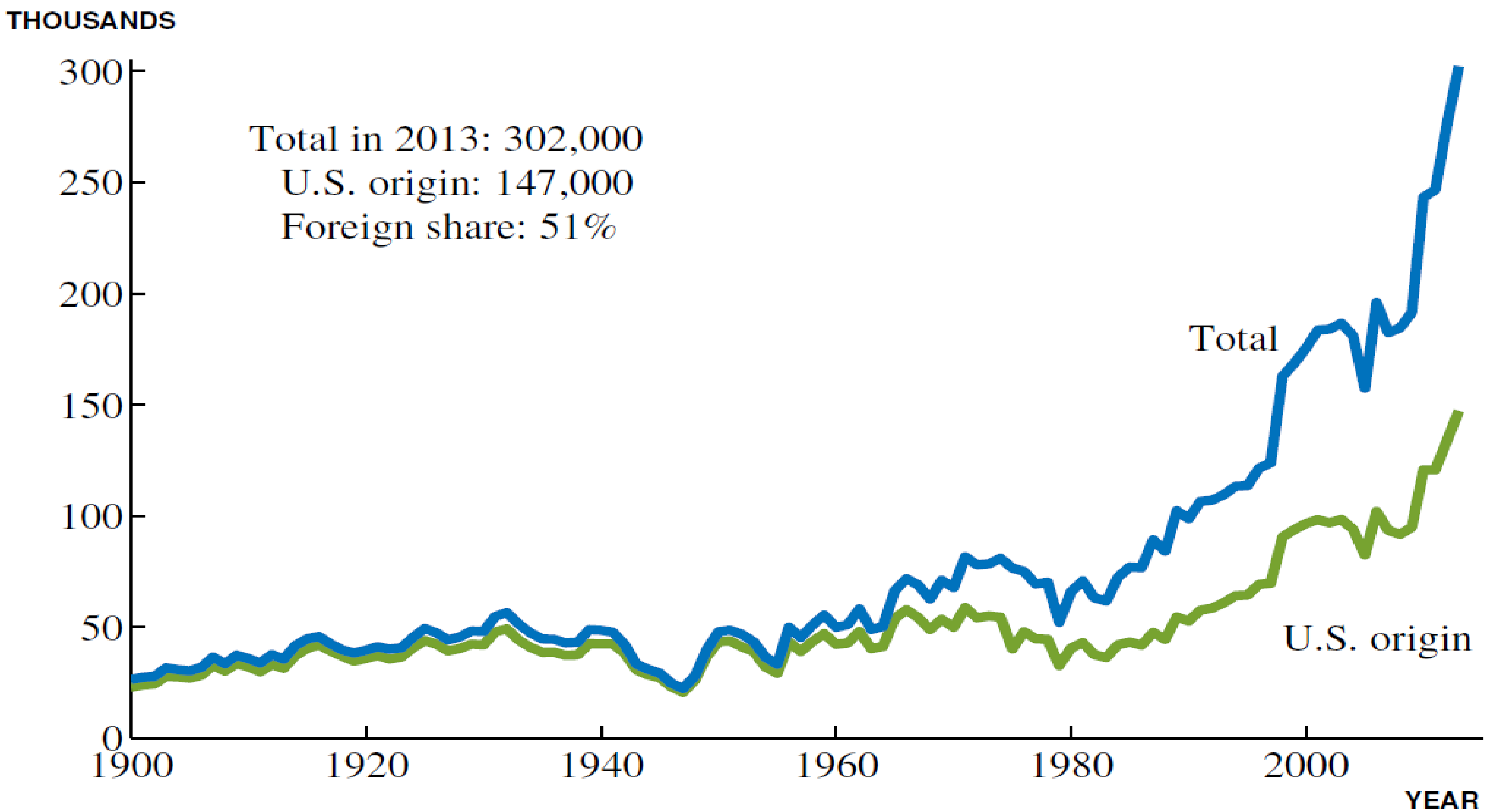


U.S. R&D Growth



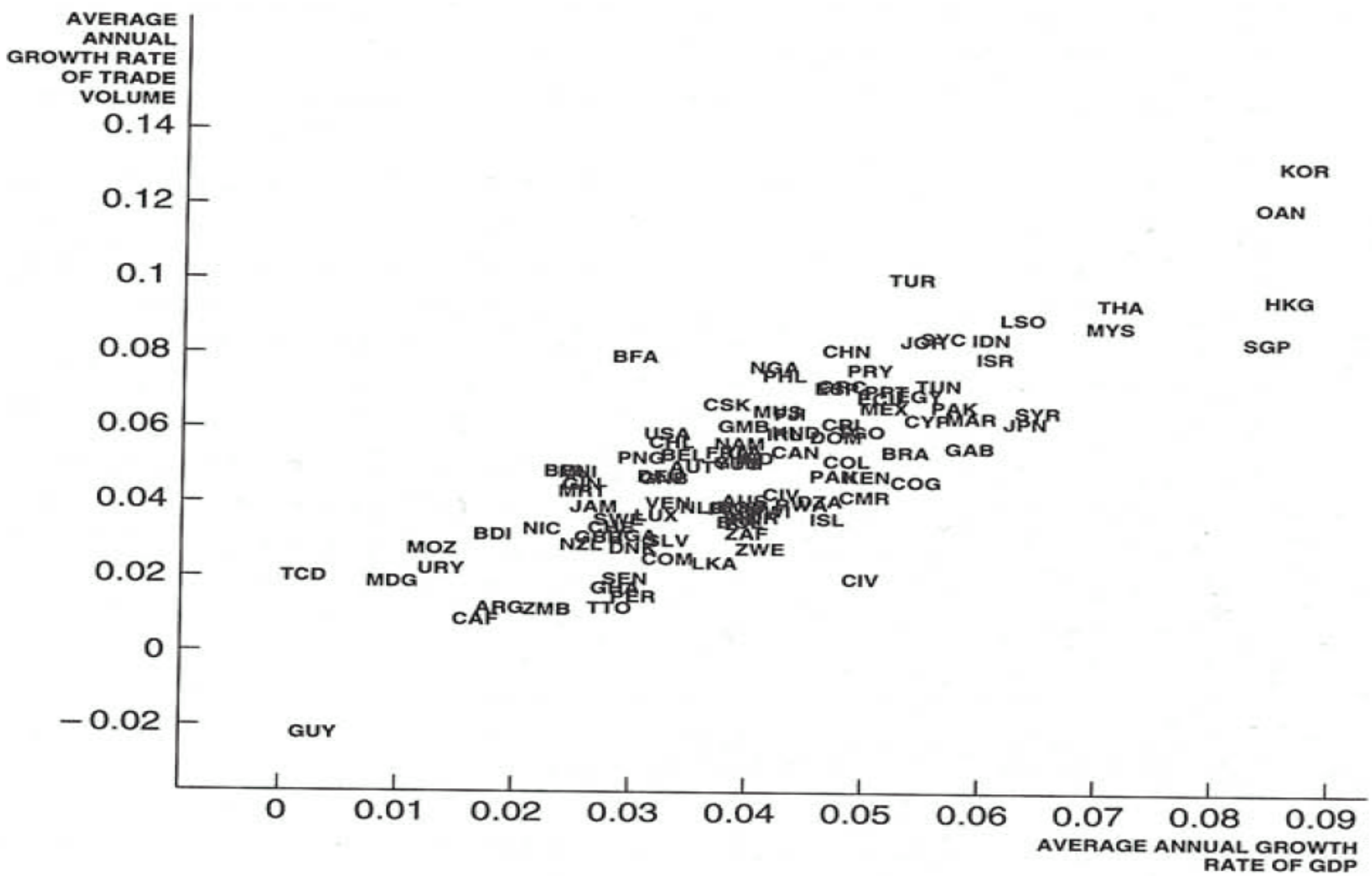
- **Government R&D expansion due to cold war**
- **Software R&D expansion since 1980**

Patents Growth (Granted by USPTO)



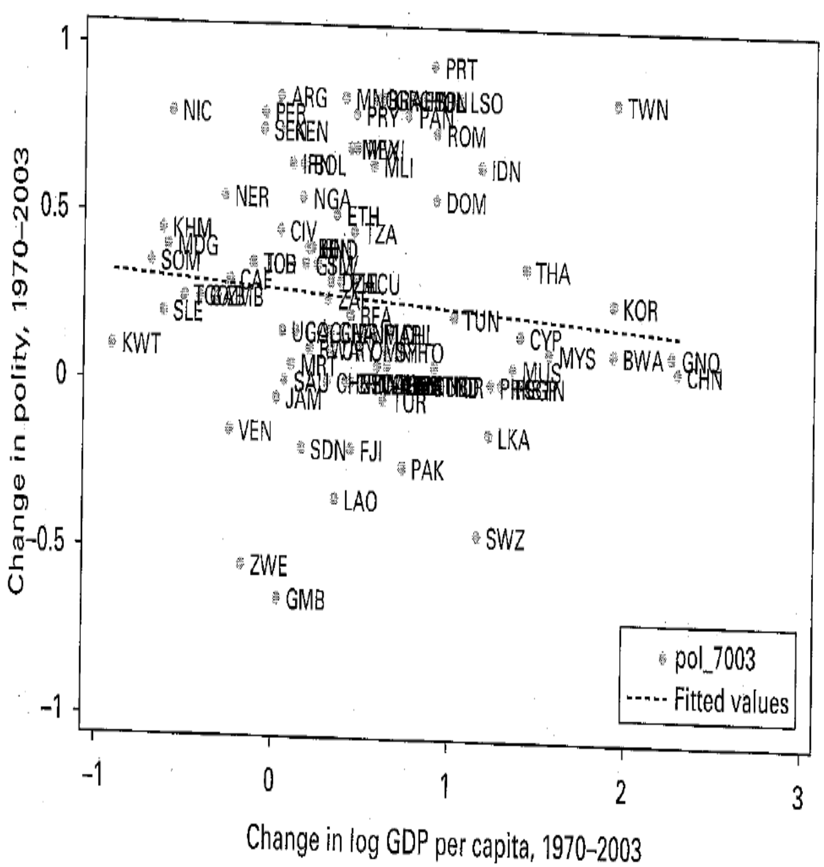
- **Patents are highly concentrated geographically:**
  - **US accounts for half**
  - **US, UK, Taiwan, South Korea, Germany, Japan together > 80%**

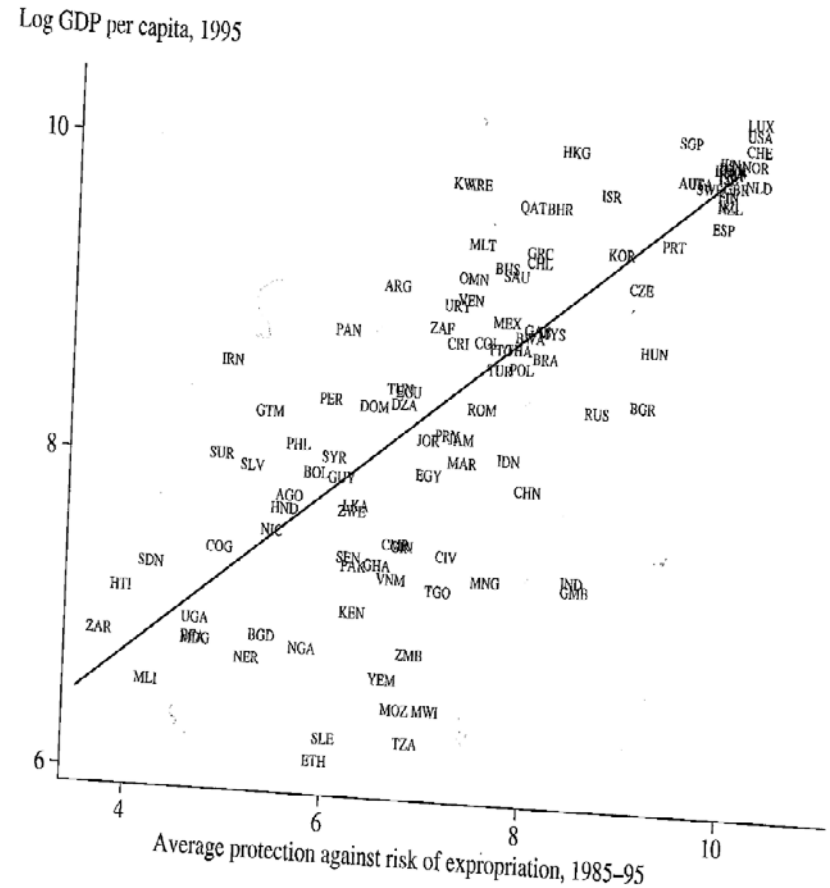
- **Trade**



- Stronger correlation than K, L and H
- Trade protection is likely detrimental

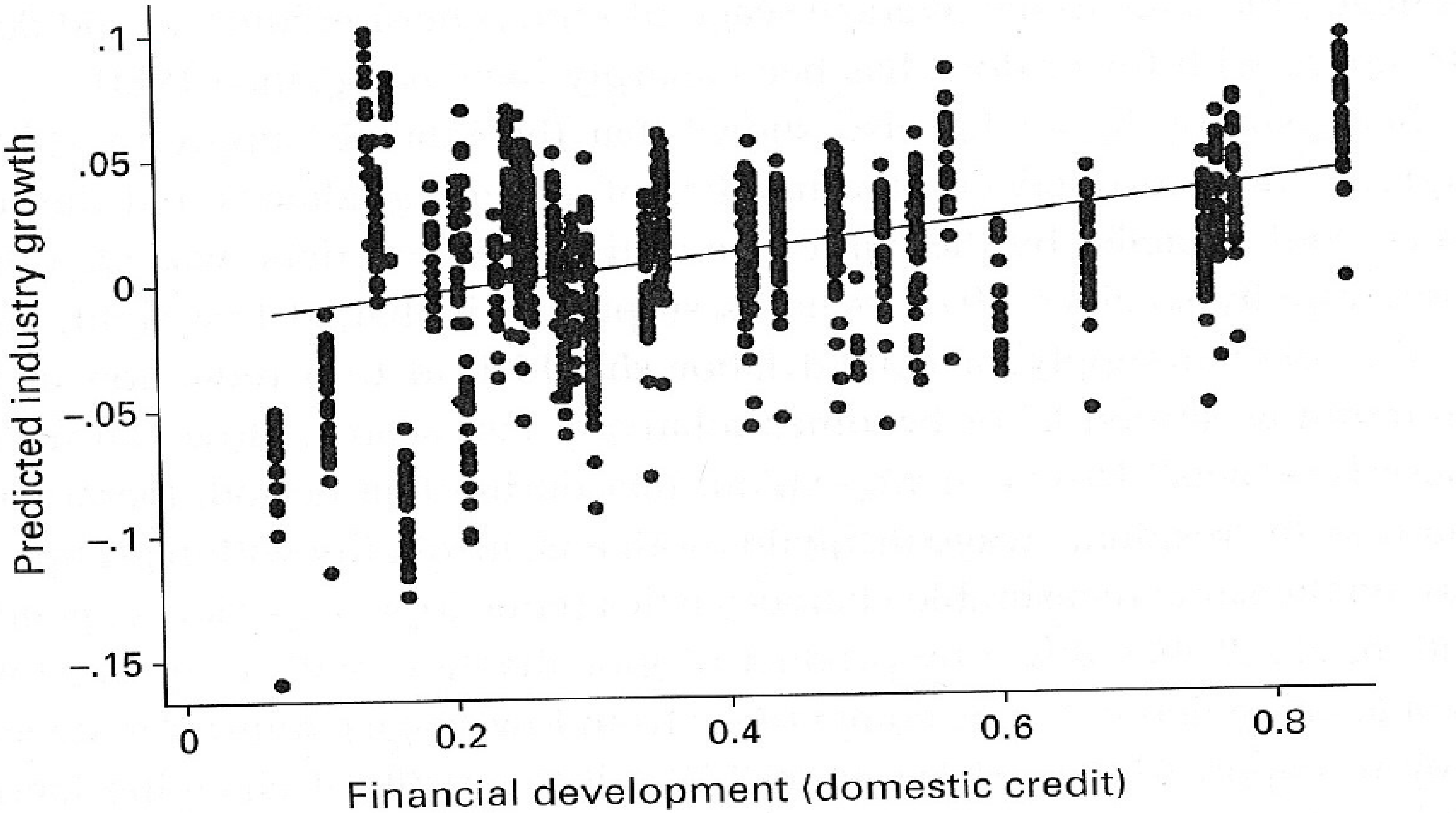
- **Institutions – democracy and IPR protection**





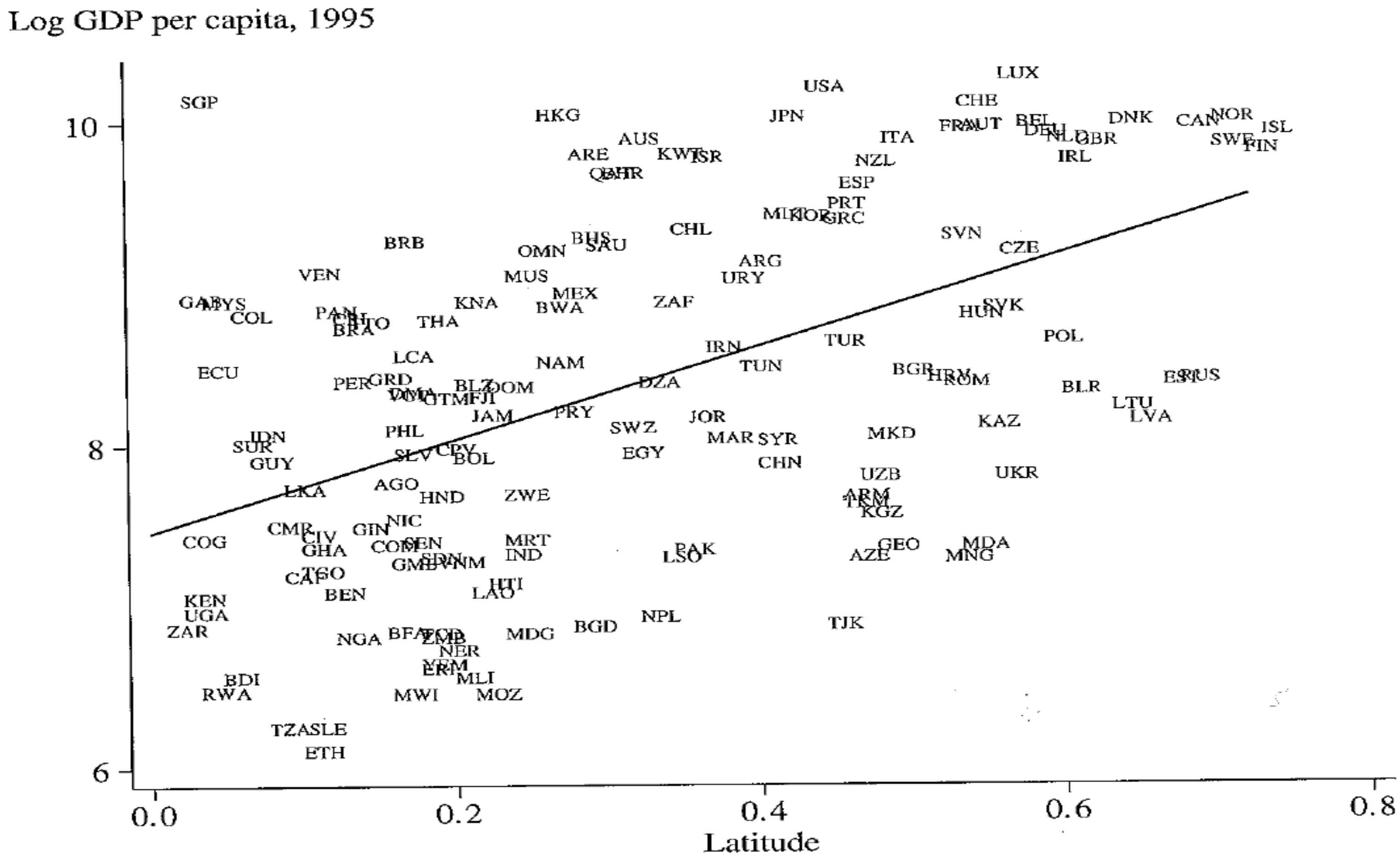
- **A tale of two systems: China vs. Taiwan (fast growth under different democracies)**
- **IPR protection promotes invention incentives**

- **Finance**



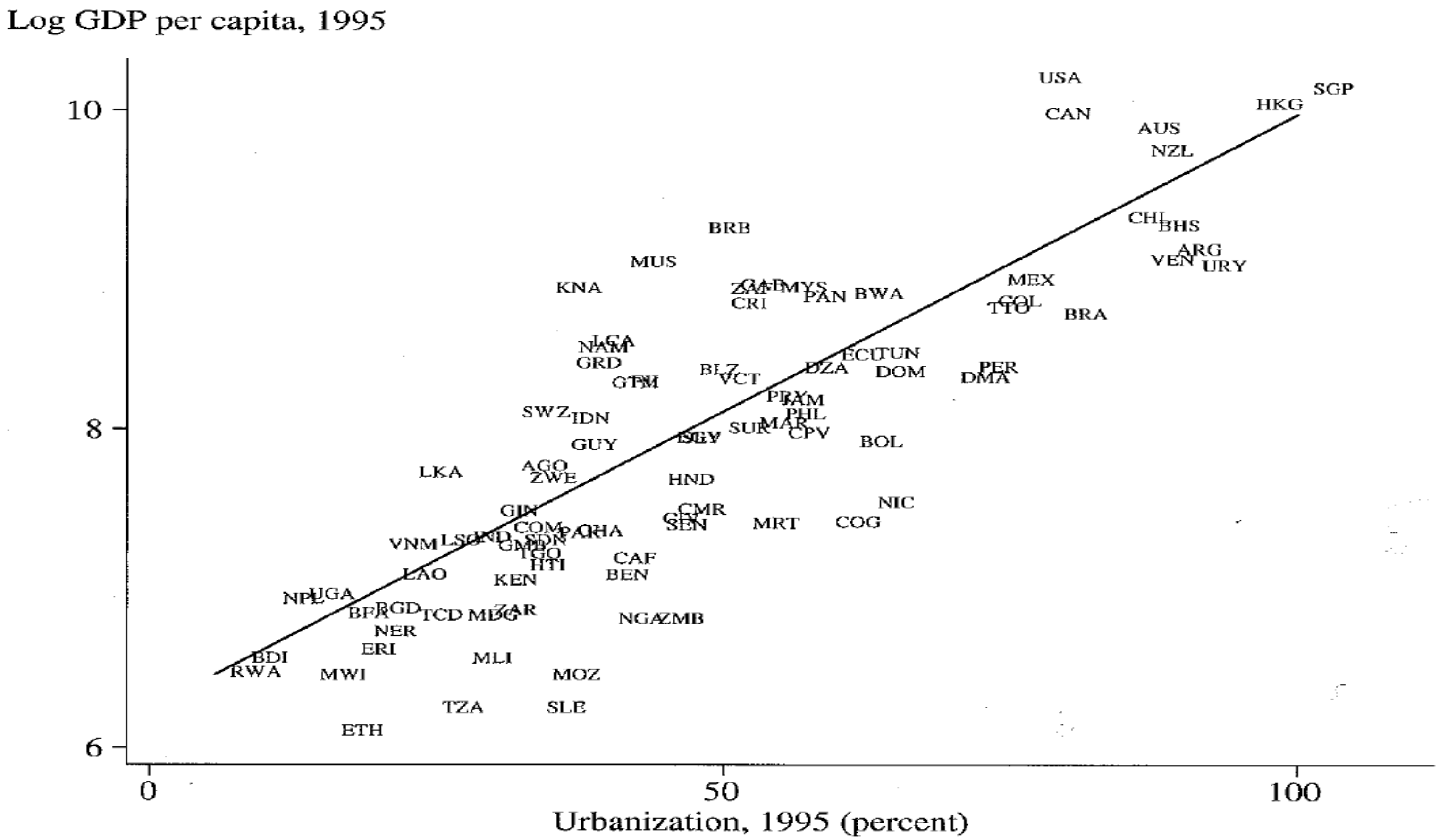
  - **Threshold effect of minimum financial development for investment purposes (at domestic credit/GDP = 0.2)**

- **Geography**



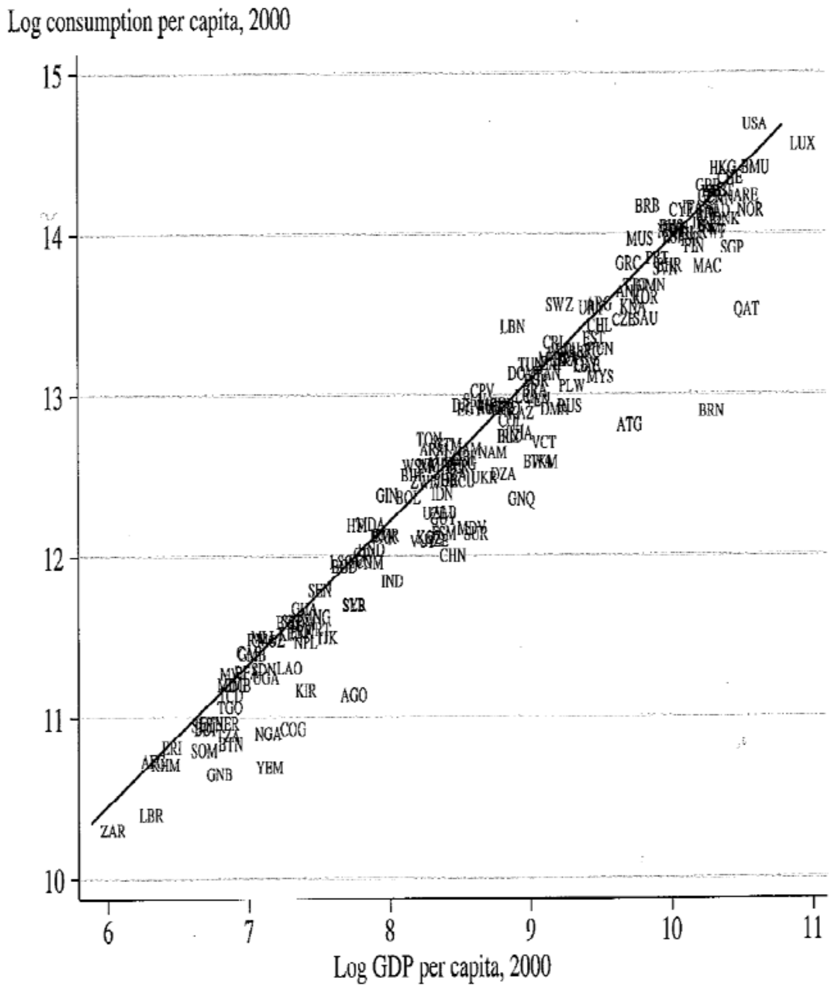
- Colonization and settlement of rulers
- Snow belt vs. sun belt

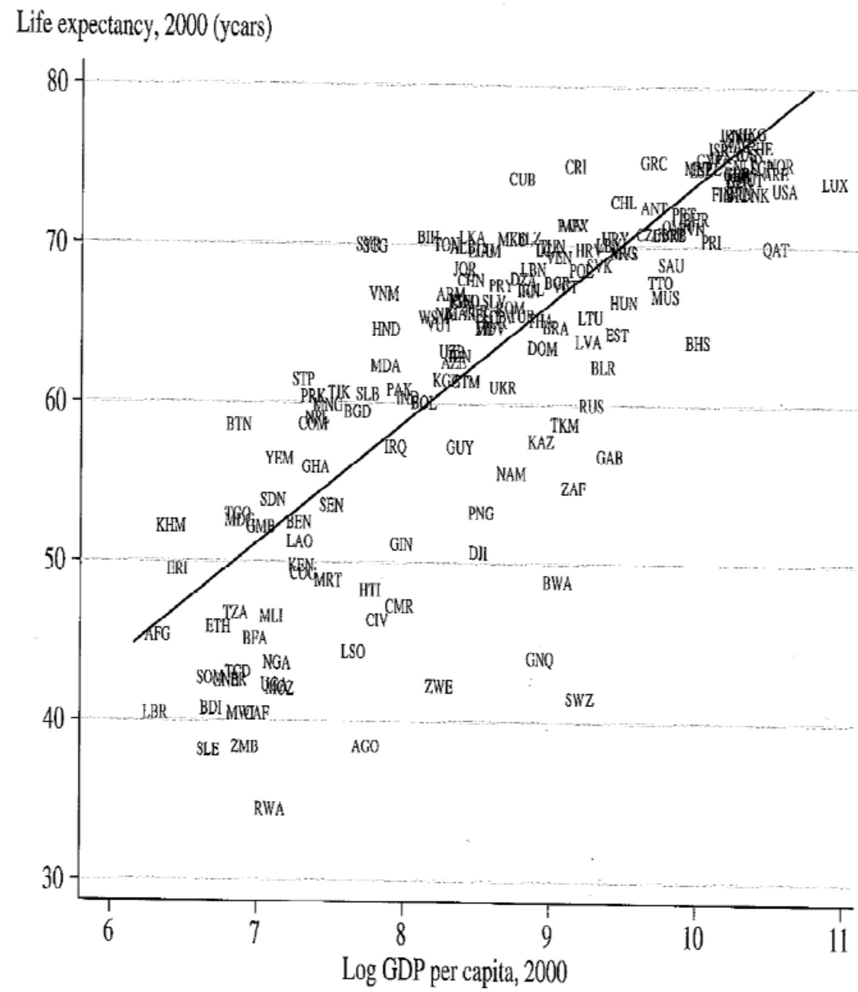
- **Urbanization**



  - o **Agglomeration economies**
  - o **Amenities and rise in services**
  - o **Reversal if using early urbanization**

3. Economic growth, consumption and life expectancy

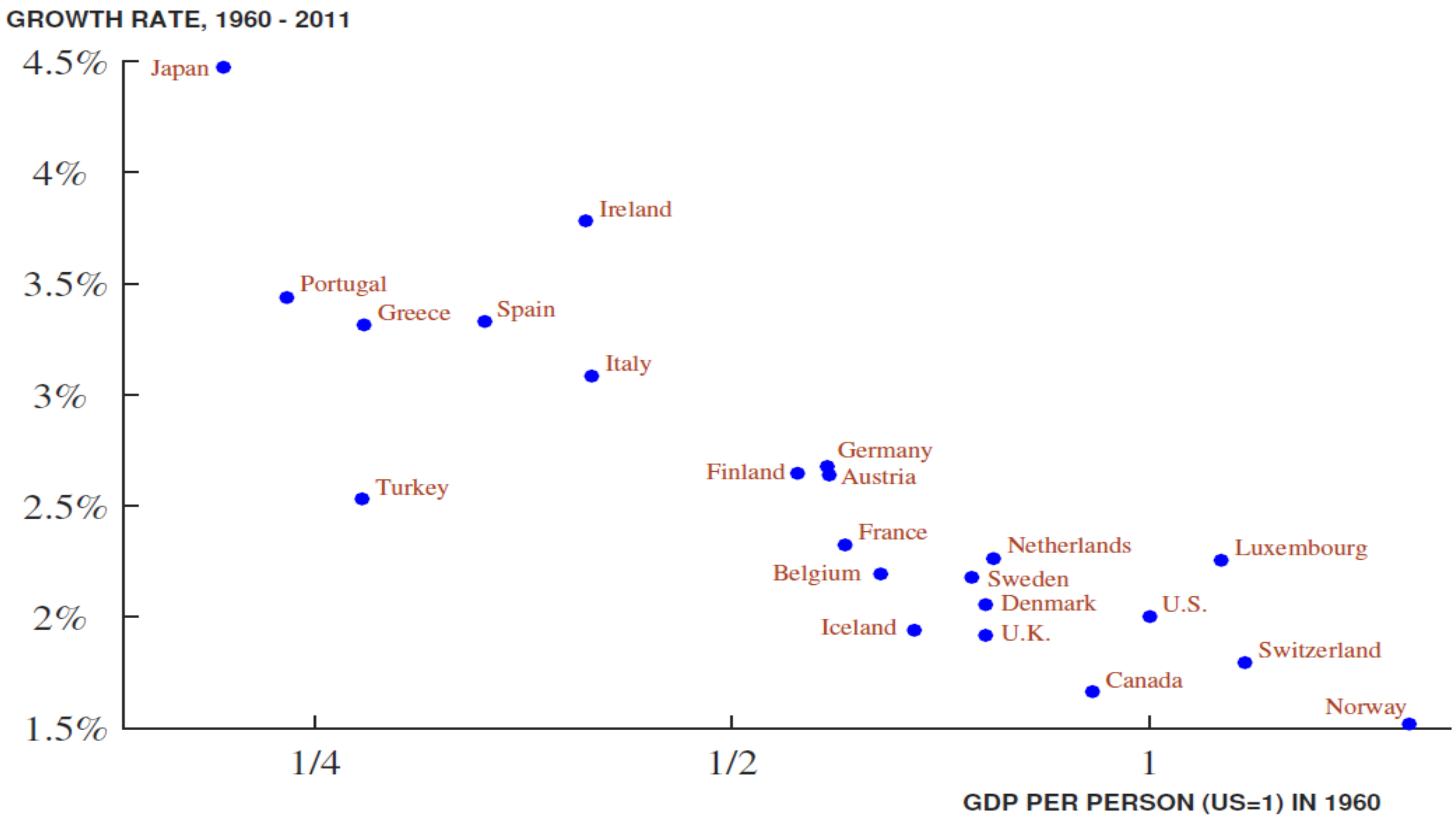




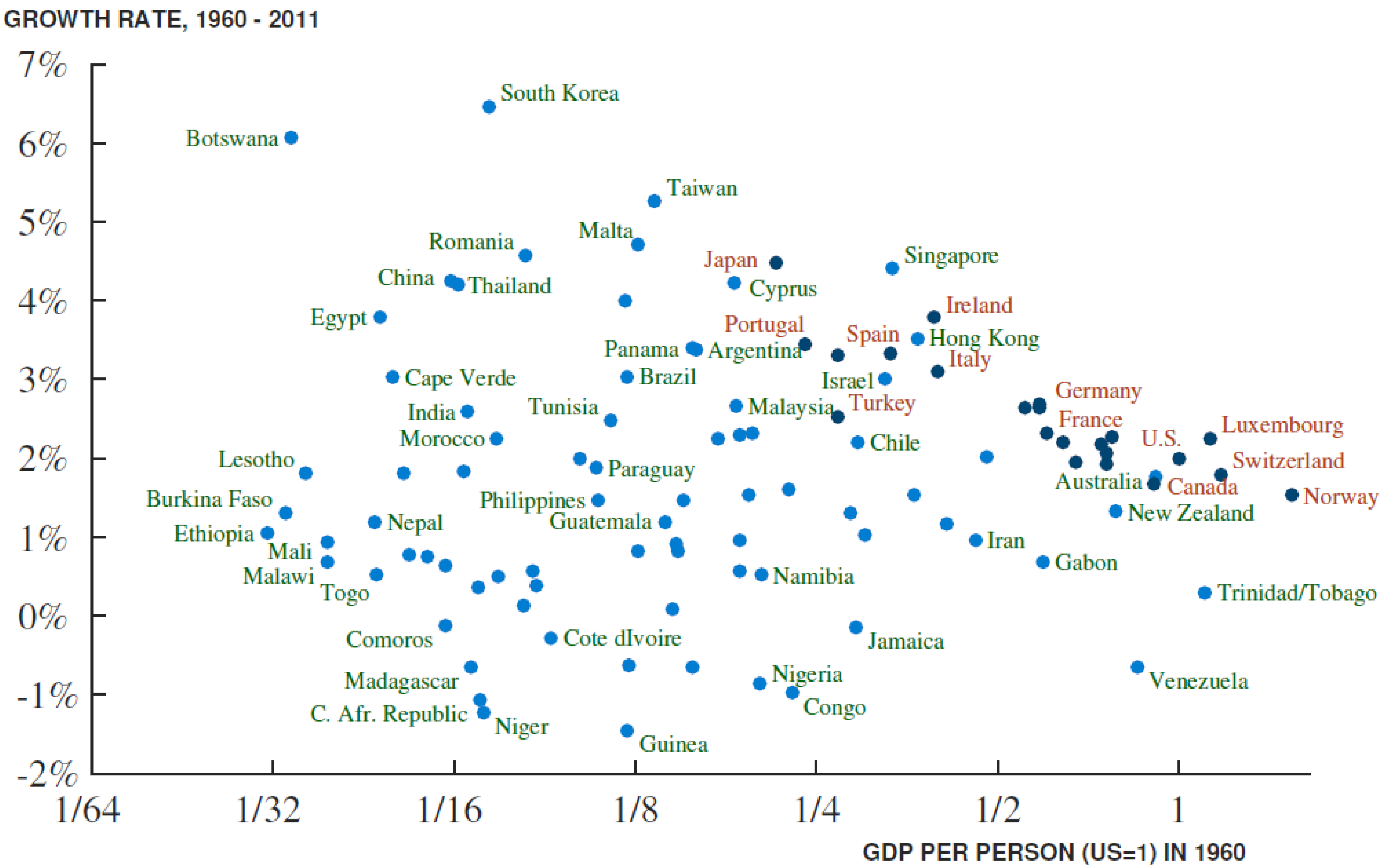
4. Have rich countries suffered growth slowdown?

- **Growth rate versus initial level of development**

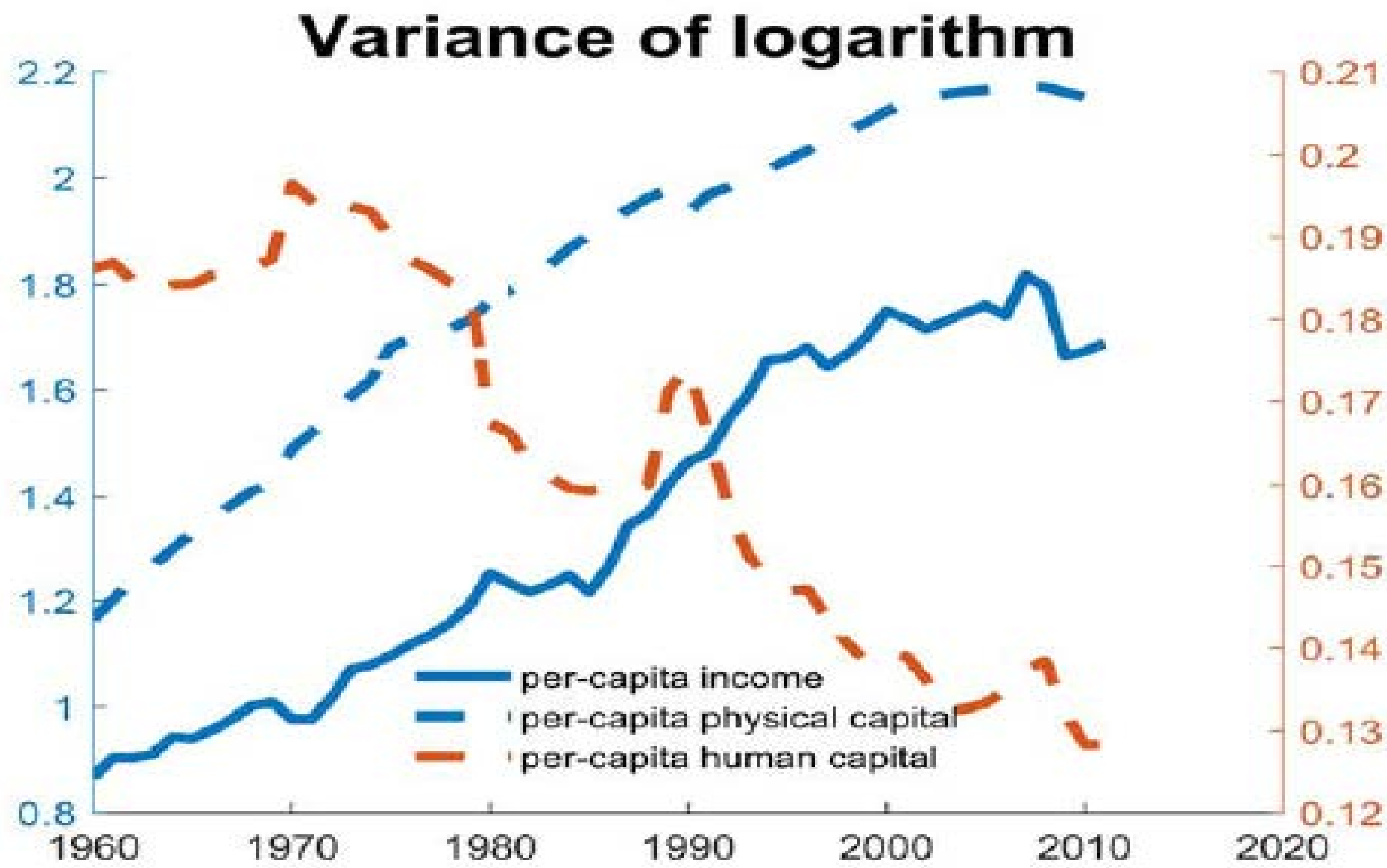
Convergence of the Rich (OECD)



Nonconvergence of the Poor



Cross-Country Disparities in Income and Factor Inputs (Wang-Wong-Yip 2016)



- **The concept of convergence and conditional convergence**

Convergence in per capita real GDP (Baumol-Barro):

  - **β-convergence: the higher the initial per capita real GDP is, the lower the per capita real GDP growth will be (β < 0)**

$$\beta\text{-convergence - } \frac{\dot{y}_i}{y_i} = \theta_i = \beta_0 + \beta y_i(0) + \ldots$$

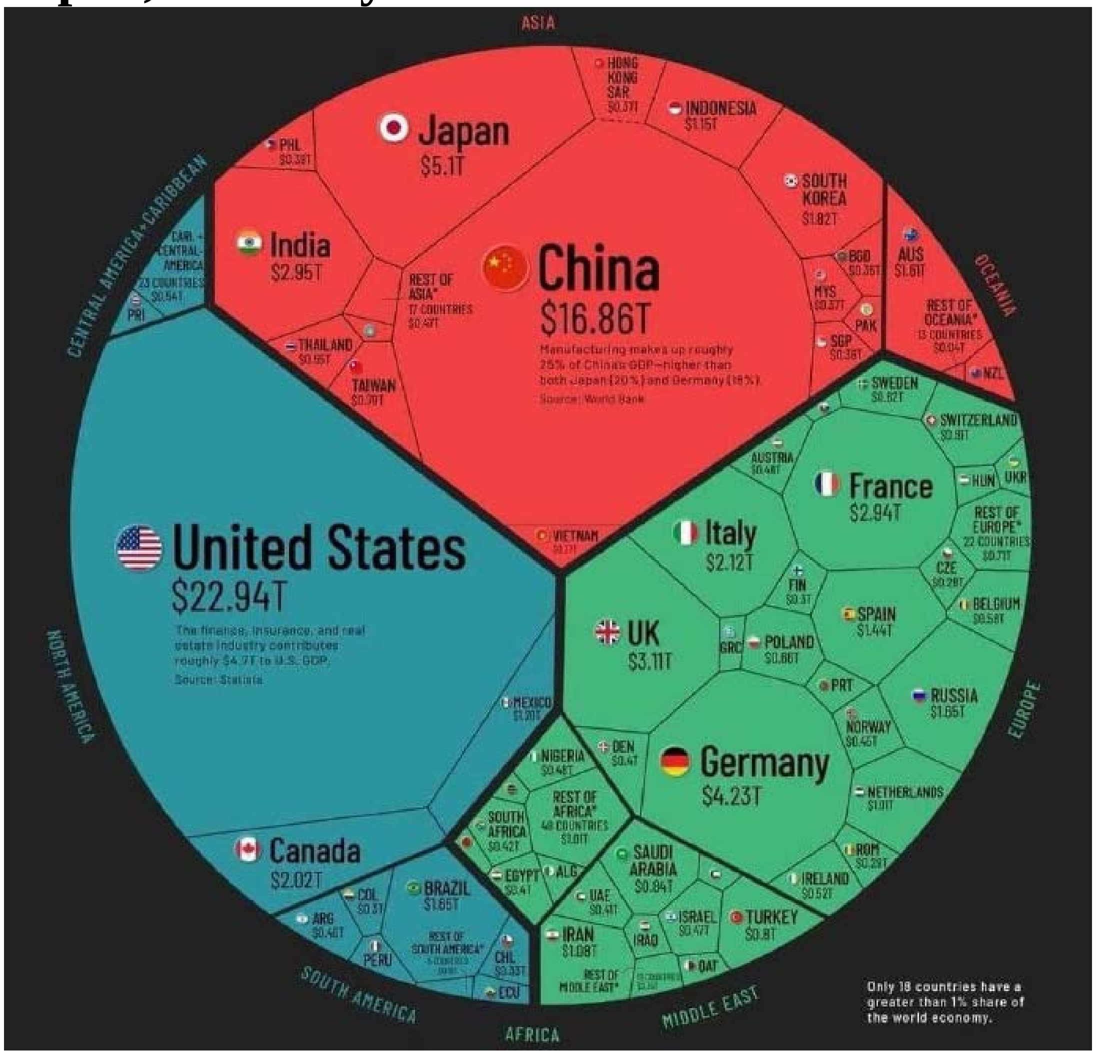
  - **σ-convergence: the cross-country per capita real GDP is decreasing over time**

$$\sigma\text{-convergence - } \frac{d}{dt}[Var(y_i)] < 0$$

Problems:

  - **Galton Fallacy (regression toward the mean)**
  - **Twin-peak hypothesis (Quah 1996)**
  - **Endogeneity problems**
  - **Measurement errors**
  - **Kitchen sink regressions**

- **World GDP by now: race to the top – only 18 countries have more than 1% share of world GDP, led by two giants, US and China, followed by Japan, Germany and UK**



C. Why Formal Theory Matters?

- **Albert Einstein: "[I]t is quite wrong to try founding a theory on observable magnitudes alone ... It is the theory which decides what we can observe."[1]**
- **Formal theory can help organizing the stylized facts observed, explaining causal relationships, offering economic predictions and drawing useful policy implications**

D. Basic Technical Tools

To build up formal dynamic general equilibrium theory, basic tools are:

- **calculus/matrix algebra, probability theory, mathematical statistics & stochastic process, basic real/functional analysis & measure theory**
- **constrained optimization methods (Lagrangian)**
- **optimal control (Maximum Principle) & stochastic control**
- **recursive methods and dynamic programming**
- **overlapping-generations (OLG) approach**
- **dynamic games**

---

[1] **While Google Translation is confusing, my own translation of the original quote with helps from my German friend is: "Logical thinking must be deductive, based on hypothetical concepts and axioms. How could we hope to be able to choose the latter in such a way that we could hope to see the proof of its consequences based on its phenomena? It is obvious that the best scenario occurs when the new basic hypotheses are suggested by the world of experience itself or by the validation of theoretical efforts in the world of experience." (Albert Einstein, *Physik und Realität*, 1935)**

Appendix: Basic Techniques

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A. Overview

A formal theory of growth/development requires the following tools:

- **simple algebra**
- **simple calculus, particularly the basic concept of differentiation**
- **basic diagrammatic analysis, including particularly:**
  - **understanding preferences & objectives**
  - **understanding technologies & constraints**
  - **demand-supply analysis**
  - **atemporal constrained optimization**
  - **intertemporal constrained optimization**

While we will overview each of the five diagrammatic tools mentioned above, we will briefly illustrate the usefulness of simple algebra and calculus in three different exercises:

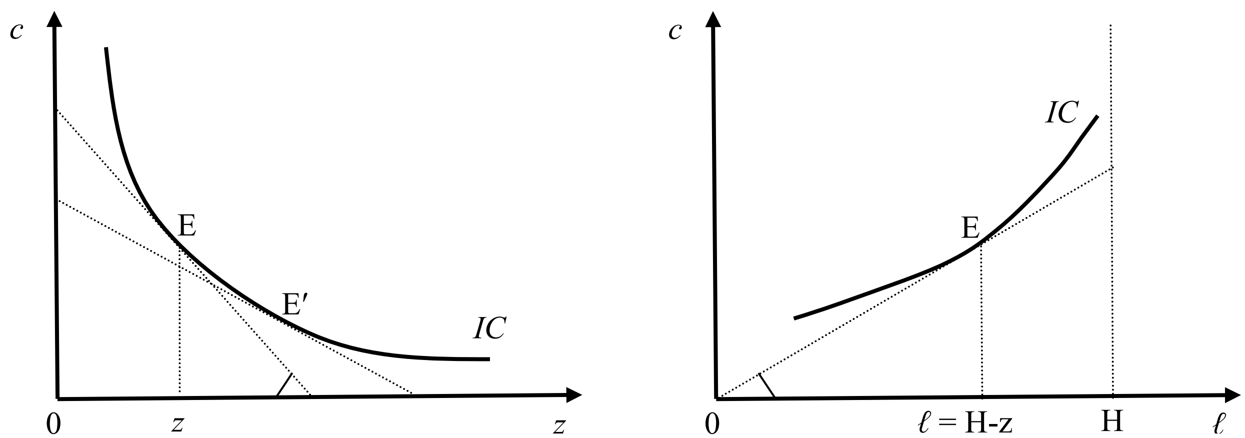
- **understanding the utility function and the production function**
- **solving equilibrium employment and real wages**
- **decomposing economic growth into productivity and factor accumulation**

B. Quick Review of Preferences and Technologies

1. Preferences

At a given point in time, a representative individual allocates her total available time (H) to work (ℓ) and leisure (z), whose preference is represented by a utility function: U = U(c, z), over a composite consumption good (c) and leisure (z = H - ℓ), satisfying:

- **strictly increasing** ($MU_c = \partial U/\partial c > 0$ and $MU_z = \partial U/\partial z > 0$)
- **diminishing MU** ($\partial MU_c/\partial c < 0$ and $\partial MU_z/\partial z < 0$)
- **diminishing MRS** ($-dc/dz|_{\text{constant U}} = MU_z/MU_c$ is decreasing in z; IC)
- **complementarity and other regularity conditions (Inada)**



2. Technologies

A prototypical one-input individual production function (backyard farming or self-employment – Robison Crusoe) is: $y = Af(\ell)$, $A > 0$, f satisfying:

- **strictly increasing** ($MPL = \partial f/\partial \ell > 0$)
- **diminishing MPL** ($\partial MPL/\partial \ell < 0$)
- **input necessity** ($f(0) = 0$)

A prototypical two-input (capital K and labor L) aggregate production function is: $Y = AF(K,L)$, with F satisfying:

- **strictly increasing** ($MPK=\partial F/\partial K > 0$ and $MPL=\partial F/\partial K > 0$)
- **diminishing MPK/MPL** ($\partial MPK/\partial K < 0$ and $\partial MPL/\partial L < 0$)
- **diminishing MRTS** ($-dK/dL|_{\text{constant } Y}$ is decreasing in L; Isoquants)
- **input necessity** ($F(0,L) = F(K,0) = 0$)
- **complementarity and other regularity conditions (Inada)**

The scaling factor, A, measures productivity, which is usually referred to as the total factor productivity (TFP). The TFP can change over time – both its level and growth rate play crucial roles in growth and development.

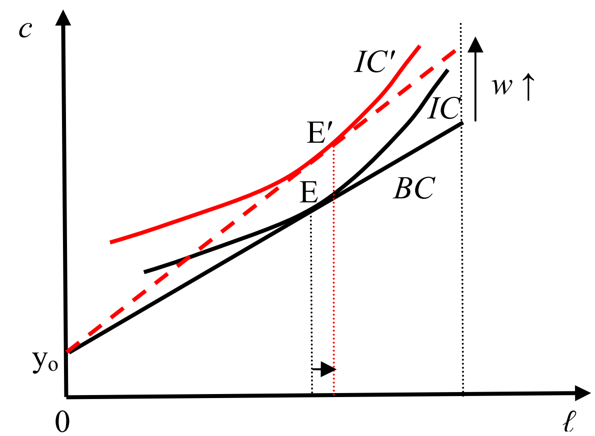
C. Demand-Supply Analysis

We will illustrate the demand-supply analysis using the labor market, based on the utility function and one-input production function presented above:

$$U = U(c,z) = U(c,H-\ell) \text{ and } y = Af(\ell)$$

Consider a simple budget constraint (BC): $c = y_o + w\,\ell$, where w = real wage rate and y_o measures non-labor income.

1. Supply of Labor

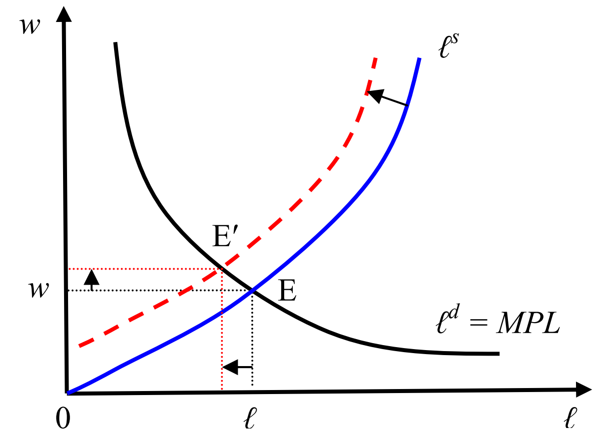


A rational individual supplies labor when she is indifferent between working and enjoying leisure: $MRS = MU_z/MU_c = w$. Under diminishing MRS and assuming normality, the labor supply is upward sloping, as shown in the diagram (higher w raises ℓ).

Note: Normality rules out backward bending labor supply, which is unlikely to a representative individual (an "average" worker).

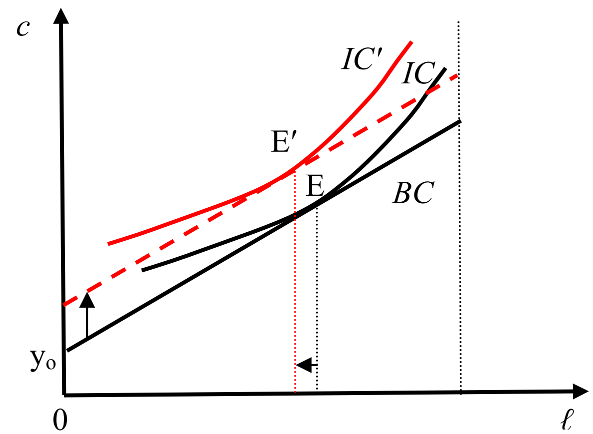
2. Demand for Labor:

In a competitive labor market, labor must be hired when its marginal product equals the wage rate prevailed: MPL = ∂f/∂ℓ = w. Under diminishing MPL, the labor demand locus is downward sloping.



3. Equilibrium

Combining labor supply and demand, we obtain equilibrium employment and wage (see point E).



4. Comparative-Static Analysis

Consider now non-labor income rises as a result of a gift (y_o higher). This discourages labor supply, thereby leading to lower employment and higher wage (shift from E to E′).

D. Decomposition of Economic Growth

Based on the two-input aggregate production function, we can decompose economic growth into productivity and factor accumulation. Such an exercise requires log calculus/total differentiation techniques ($\dot{x} = dx/dt$):

$$\ln(x \cdot y) = \ln(x) + \ln(y), \quad \ln(\frac{x}{y}) = \ln(x) - \ln(y), \quad \frac{d\ln(x)}{dt} = \frac{1}{x}\frac{dx}{dt} = \frac{\dot{x}}{x}$$

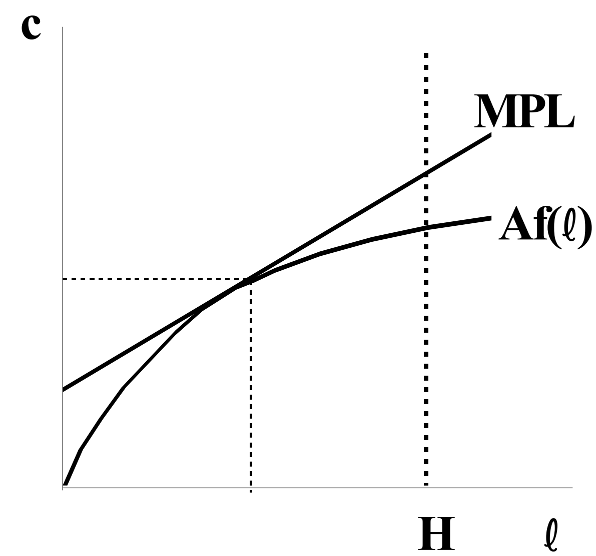
$$\frac{d\ln(x \cdot y)}{dt} = \frac{d\ln(x)}{dt} + \frac{d\ln(y)}{dt} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y}, \quad \frac{d\ln(x/y)}{dt} = \frac{d\ln(x)}{dt} - \frac{d\ln(y)}{dt} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y}$$

We can now conduct the decomposition exercise:

- **Production function: $Y = AF(K,L) = A\,K^{\alpha}\,L^{1-\alpha}$**
- **Taking natural log: $\ln(Y) = \ln(A) + \alpha\ln(K) + (1-\alpha)\ln(L)$**
- **Total differentiating with respect to t:** $\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}$.

Thus, economic growth can be decomposed into a TFP component ($\dot{A}/A$), a capital accumulation component ($\dot{K}/K$) that is weighted by α, and an employment enhancement component ($\dot{L}/L$) that is weighted by 1-α.

E. Atemporal Optimization (Robison Crusoe)



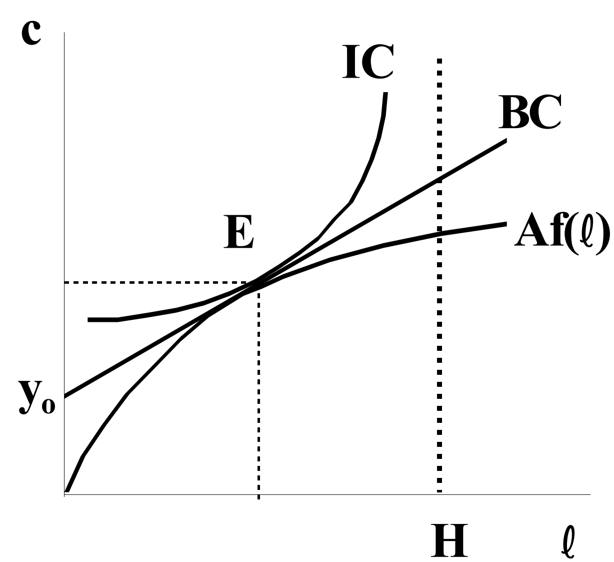
1. Self-Employment Labor Supply Decision

MRS = $(\partial U/\partial z)/(\partial U/\partial c)$ = w (implicit wage)

2. Self-Employment Labor Demand Decision

MPL = $\partial f/\partial \ell$ = w (implicit wage)

3. Atemporal Self-Employment Equilibrium



- **Goods market equilibrium**

  $c = y = Af(\ell)$ (with $y_o = Af(\ell) - MPL\cdot\ell$)

- **Equilibrium Conditions**

  MRS = w = MPL (points E)

F. Intertemporal Optimization: Two-Period Fisherian Model

1. Intertemporal Consumption Choice

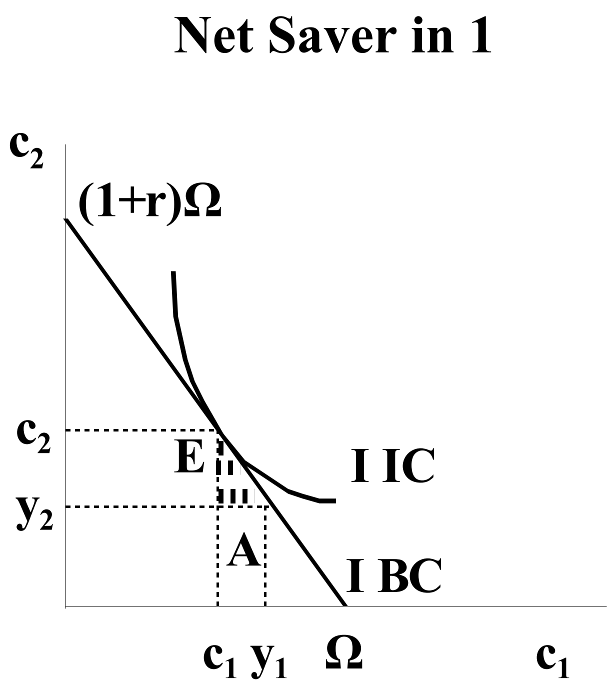
- **Lifetime Utility: $V = U(c_1 , c_2)$, strictly increasing and satisfying diminishing MU, diminishing MRS and other regularity conditions to yield well-behaved intertemporal indifference curves (IIC) – a commonly used lifetime utility assumes time-additive periodic utility with constant time discounting: $V = u(c_1) + \beta\, u(c_2)$, with $\beta = 1/(1+\rho) < 1$.**

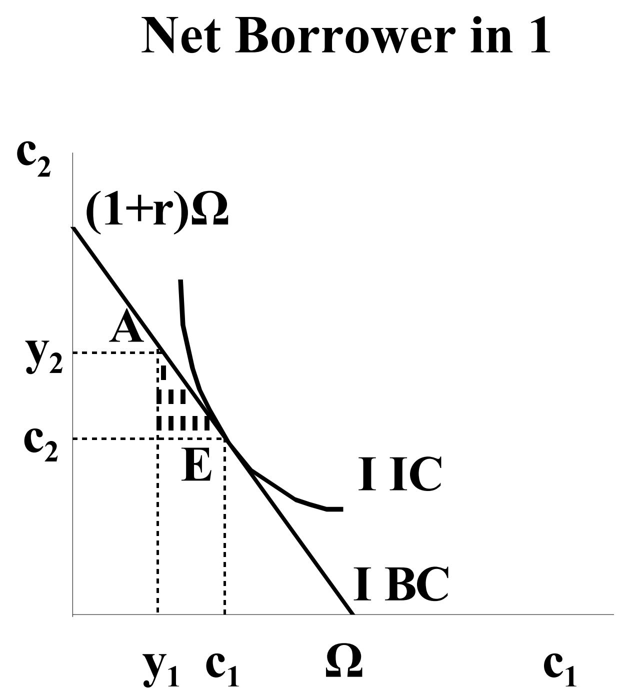
- **Lifetime Budget: Since lifetime wealth (Ω) = current income + present value of future income (based on a real rate of interest, r, and the initial real bond holding b_0), the intertemporal budget constraint (IBC) is:
$c_1 + c_2 /(1+r) = b_0 + y_1 + y_2 /(1+r) = \Omega$**

2. Optimized Decision

- **Marginal rate of intertemporal substitution:
$MRS = (\partial U/\partial c_1)/(\partial U/\partial c_2) = 1 + r$**

- **Diagrammatic illustration:**





A = autarchy point; E = optimized decision
Intertemporal trade triangles => net saving/borrowing decisions

- **Mathematically, we have:**
  - $b_0-(y_2-c_2)/(1+r)<0$ **=> net saver in 1 (2 = retirement)**
  - $b_0-(y_2-c_2)/(1+r)<0$ **=> net borrower in 1 (1 = childhood/studentship)**

3. From 2-Period to Infinite Horizon

To better approximate the real world, we extend the 2-period model to infinite horizon and, for simplicity, allow the representative agent to live forever (finite lives "love thy children" with positive bequest motives):

- **Lifetime utility:** $V = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \ldots = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$
- **Lifetime budget:** $c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + \ldots = b_0 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} + \ldots$,

  or, $\sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^{t-1} c_t = b_0 + \sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^{t-1} y_t$

4. From Discrete Time to Continuous Time

The continuous-time counterpart is usually more parsimonious:

- **Lifetime utility:** $V = \int_0^\infty u(c(t)) e^{-\rho t} dt$
- **Lifetime budget (under constant r):** $\int_0^\infty c(t) e^{-rt} dt = b_0 + \int_0^\infty y(t) e^{-rt} dt$, **or, at each point in time,** $\dot{\Omega} = r\Omega + y(t) - c(t)$, **with** $\Omega(0) = b_0$

Appendix to the
Basic Techniques
Appendix

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* This note is provided for your “bedtime reading” to facilitate better understand of constrained optimization and continuous-time intertemporal optimization.

A. Constrained Optimization and Lagrangian

- **The constrained optimization problem (COP):**
  Reward function: R(x,y), strictly increasing and strictly concave in each argument and quasi-concave in (x,y), i.e., $R_x > 0$, $R_y > 0$, $R_{xx} < 0$, $R_{yy} < 0$, $R_{xx}R_{yy} - (R_{xy})^2 \geq 0$
  Constraint: $g(x,y) = z$, where the constraint set is convex
- **The Lagrangian Method (λ = Lagrangian multiplier):**
  $$\mathcal{L}(x,y,\lambda) = R(x,y) + \lambda\,[z - g(x,y)]$$
  First-order necessary conditions ($\partial\mathcal{L}/\partial x = \partial\mathcal{L}/\partial y = \partial\mathcal{L}/\partial\lambda = 0$):
  $$R_x - \lambda g_x = 0, \qquad R_y - \lambda g_y = 0, \qquad z - g(x,y) = 0$$
  The second-order sufficient conditions are usually met when the reward function is quasi-concave and the constraint set is convex
- **Example: Let the reward function be a standard utility function and the constraint be a standard budget constraint:**
  max $U(x,y)$ subject to $p_x x + p_y y = I$
  Quasi-concavity of U => indifference curves convex toward the origin
  Then, $\mathcal{L} = U(x,y) + \lambda\,(I - x - py)$ and the first-order conditions are:
  $$U_x - \lambda p_x = 0, \qquad U_y - \lambda p_y = 0, \quad I - p_x x - p_y y = 0$$

B. Intertemporal Optimization in Continuous Time: Optimal Control

Optimal control solves continuous-time dynamic constrained optimization problem (DCOP). Rather than using the Lagrangian, we employ current-value Hamiltonian ($\mathcal{H}$)

- **The DCOP with one control (x) and one state (y):**

  max PDV of lifetime reward = $\int_0^\infty R(x,y)e^{-\rho t}dt$

  s.t. $\dot{y} = g(x,y)$, $y(0) = y_0 > 0$ **(initial condition)**
- **Pontryagin maximum principle (λ = costate):**

(Step 1) Set up current-value Hamiltonian

$$\mathcal{H}(x,y,\lambda) = R(x,y) + \lambda\, g(x,y)$$

(Step 2) Derive first-order necessary condition(s) w.r.t. control(s):

$$R_x + \lambda g_x = 0$$

(Step 3) Obtain Euler equation(s) w.r.t. state(s):

$$\dot{\lambda} = \rho\lambda - \partial\mathcal{H}/\partial y = \lambda(\rho - g_y)$$

Notes:

- The condition, $\dot{y} = \partial \mathcal{H}/\partial \lambda = g(x,y)$, is redundant
- The sufficient conditions:
  - maximized Hamiltonian $\mathcal{H}(x^*(y),y)$ is strictly concave in y
  - a terminal (Transversality) condition is met: $\lim_{t\to\infty} \lambda y e^{-\rho t} = 0$
  - in all models considered, these sufficient conditions are always met; we thus ignore for the sake of brevity

Example: Let the reward function be u(c) and $g(c,k) = f(k) - \delta k - c$.

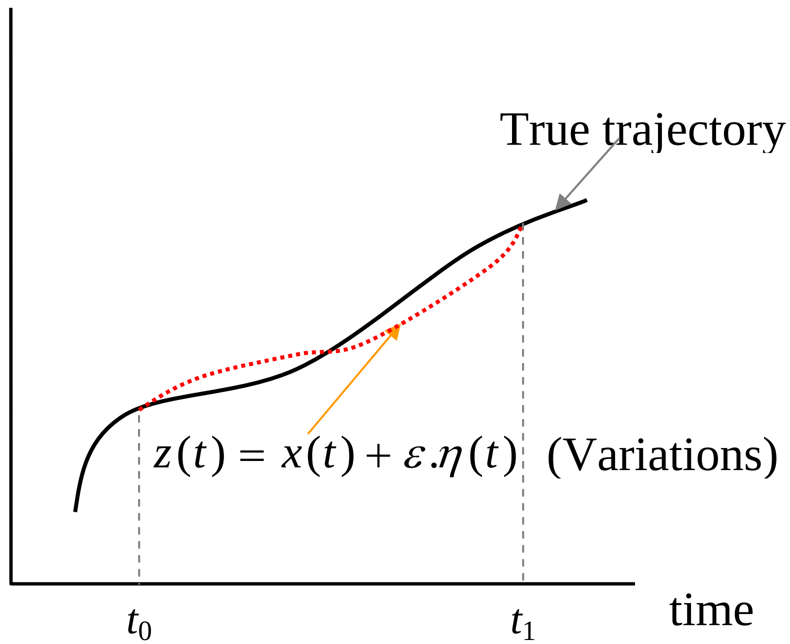
(Step 1) Current-value Hamiltonian: $\mathcal{H}(c,k,\lambda) = u(c) + \lambda\, [f(k) - \delta k - c]$

(Step 2) First-order condition: $u_c - \lambda = 0$

(Step 3) Euler equation: $\dot{\lambda} = \lambda(\rho + \delta - f_k)$

- **For multiple n controls and m states, there will be n first-order conditions and m Euler equations**

- **For those interested, it is noted that the Euler equation can be derived using the "Fundamental Lemma" of Calculus of Variation (CV)**



- **Calculus of variation: minimize deviations from the true trajectory:**

$$R(\varepsilon) = \int_{t_0}^{t_1} r(z,\ \dot{z},\ t)dt = \int_{t_0}^{t_1} r(x+\varepsilon\eta,\ \dot{x}+\varepsilon\dot{\eta},\ t)dt$$

$$\Rightarrow\ 0 = \frac{dR(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon\to 0} = \int_{t_0}^{t_1} \frac{\partial r}{\partial x}\eta\, dt + \int_{t_0}^{t_1} \frac{\partial r}{\partial \dot{x}}\dot{\eta}\, dt$$

- **From integration by part:**

$$\frac{dR(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon\to 0} = \int_{t_0}^{t_1} \frac{\partial r}{\partial x}\eta dt + \eta\frac{\partial r}{\partial x}\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \eta\frac{d}{dt}\frac{\partial r}{\partial \dot{x}}dt \;\; = \int_{t_0}^{t_1}\left[\frac{\partial r}{\partial x} - \frac{d}{dt}\frac{\partial r}{\partial \dot{x}}\right]\eta(t)dt = 0$$

- **For any $\eta(t)$, the above equation must equal to 0, implying: $\frac{\partial r}{\partial x} = \frac{d}{dt}\frac{\partial r}{\partial \dot{x}}$.**

  This is the Fundamental Lemma of Calculus of Variation (FLCV).

- **Consider discounting and write the present-value Hamiltonian in the conventional CV form: $\mathcal{H}^*(c,k,\dot{k},\lambda) = e^{-\rho t}\{u(c) + \lambda[f(k)-\delta k-c-\dot{k}]\} = e^{-\rho t}\mathcal{H}$, where $\lambda=\lambda^* e^{\rho t}$ and $\partial\mathcal{H}^*/\partial\lambda^*=0$ by construction**

- **Regarding *r* as $\mathcal{H}^*$ and *x* as k, we can then apply FLCV to obtain:**

  $\frac{\partial H^*}{\partial k} = \frac{d}{dt}\frac{\partial H^*}{\partial \dot{k}} = \frac{d}{dt}(-\lambda^*)$, or, $\frac{\partial H}{\partial k} = -(\dot{\lambda} - \lambda\rho)$, which yields the Euler equation associated with k ($\dot{\lambda} = \lambda\rho \; - \partial\mathcal{H}/\partial k$)