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Alternative Approaches to Money and Growth

BEGINNING WITH THE CLASSIC PAPERS of Tobin (1965) and Sidrauski (1967), the analysis of the effects of anticipated inflation on capital accumulation has been one of the central issues in monetary growth theory. In their well-cited work, Dornbusch and Frenkel (1973) provided a comprehensive survey and insightful discussion of this argument. Over the past fifteen years, there have been substantial changes in developing general equilibrium models of money. As a complement to Dornbusch and Frenkel's (1973) article, our paper reexamines the issue of money and growth using three recently developed approaches: the money-in-the-utility-function (MIUF) approach, the cash-in-advance (CIA) approach, and the transactions-costs (TC) approach. We then establish a *qualitative equivalence* between alternative approaches. Different from the Feenstra (1986) functional equivalence analysis, we focus on the policy implications of various theoretical constructs by only imposing restrictions to obtain qualitatively identical comparative statics results. Such an equivalence would provide us with robustness of the underlying general equilibrium framework yielding consistent theoretical predictions and offering intuition for values of model parameters researchers can pick up.

By and large, there is no consistent theoretical prediction of the effects of the money growth rate on economic aggregates, nor is the empirical significance of these real effects agreed upon. The inconsistency of theoretical conclusions is due in principle to the different ways in which money is introduced into the system. When

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money enters directly into the utility function and interacts with an elastic leisure demand (Brock 1974), money is generally nonsuperneutral. When money is regarded as a substitute for capital (Tobin 1965), higher monetary growth enhances capital accumulation. When money is required for purchasing capital goods (Stockman 1981), higher anticipated inflation decreases steady-state real balances and capital stock, and hence a *reversed* Tobin effect emerges. When money is introduced through a pecuniary transactions cost function (Dornbusch and Frenkel 1973), the real effect of money is ambiguous. When money serves as a transaction device through a shopping-time technology (Saving 1971; Kimbrough 1986), the theoretical predictions become more clear-cut. These inconsistent comparative statics results seem incompatible with the functional equivalence analysis in which one approach is equivalent to others under some conditions (McCallum 1983; Feenstra 1986). It is, therefore, desirable to investigate the sources of such theoretical inconsistency.

In this paper, we consider a monetary economy with *both* endogenous labor and endogenous capital, in contrast to Stockman (1981) and Saving (1971), such that a generalized cash-in-advance model and a generalized transactions-costs model can be constructed as compared with a Brock-like money-in-the-utility-function model. In general, the establishment of a functional equivalence involves complex redefinition of choice variables and thus may result in changes in the nature of each model. To circumvent these problems, we develop a weaker concept—a qualitative equivalence. Specifically, we derive conditions (or restrictions) imposed on the structure (or functional forms) of various approaches such that all approaches yield exactly identical comparative statics results (in signs). We find that such conditions exist amongst the three aforementioned alternatives: (i) Pareto complementarity between consumption and leisure; (ii) Pareto complementarity between consumption and money; (iii) Pareto substitutability between leisure and money; and (iv) weakly dominant consumption effect of money growth compared to the real balance effect.[1] Under these conditions, higher monetary growth results in higher steady-state leisure, but leads to lower steady-state capital, labor, real money balances, consumption, and welfare.

1. THE MONEY-IN-THE-UTILITY-FUNCTION APPROACH

We follow Brock (1974) to allow for direct utility interactions between leisure and real balances. The representative agent's optimization problem is

$$\max \int_0^\infty W(c(t),\ell(t),m(t))e^{-\rho t}dt \tag{1}$$

$$\text{s.t. } c(t) + \dot{k}(t) + \dot{m}(t) = f(k(t), \ell(t)) - nk(t) - (\pi(t) + n)m(t) + \tau(t) \ , \tag{2}$$

[1]The detailed proof of the results is available upon request.

where c and m are (per capita) consumption and real money balances, respectively; k and ℓ denote (per capita) capital stock and labor effort, respectively; ρ, n, and π are (constant) rates of time preference, population growth, and inflation, respectively; and τ denotes lump-sum real money transfer payments. The instantaneous utility function W is assumed to be well-behaved, satisfying $W_c > 0$, $W_\ell < 0$, $W_m > 0$, $W_{cc} < 0$, $W_{\ell\ell} < 0$, $W_{mm} < 0$, and the Inada conditions. The instantaneous production function f is assumed to be well-behaved and to exhibit constant returns, satisfying $f_k > 0, f_\ell > 0, f_{kk} < 0, f_{\ell\ell} < 0, f_{k\ell} > 0$, and the Inada conditions.

Given (constant) money growth rate μ, money market equilibrium requires $\tau = \mu m$ and $\dot{m} = (\mu - \pi - n)m$. In the steady state, $\dot{c} = \dot{\ell} = \dot{m} = \dot{k} = 0$ and hence $\pi = \mu - n$. Straightforward application of Pontryagin's Maximum Principle and manipulation of the resulting first-order conditions yield

$$f_k(k, \ell) = \rho + n \tag{3}$$

$$W_c(c, \ell, m) f_\ell(k, \ell) = -W_\ell(c, \ell, m) \tag{4}$$

$$W_m(c, \ell, m) = (\rho + \mu) W_c(c, \ell, m) \,. \tag{5}$$

(3) is the modified golden rule equation determining optimal steady-state capital-labor ratio (k/ℓ), while (4) and (5) are the fundamental no-arbitrage conditions of ℓ and m that equate marginal benefits of ℓ and m with their marginal costs, respectively. The equation system (3)–(5) characterizes the steady-state solution of (k, ℓ, m) when c is rewritten as in terms of (k, ℓ) by using the steady-state budget constraint: $c = f(k, \ell) - nk$.

The comparative statics results are summarized in Table 1. Suppose money is separated from consumption and labor in the utility function in such a way that $W_{cm} = W_{\ell m} = 0$. Then $dk/d\mu = d\ell/d\mu = 0$. If we further assume $W_{c\ell} < 0$, it is trivial to show that $dm/d\mu < 0$. so the Sidrauski (1967) superneutrality result emerges.

In general, the comparative statics results depend crucially on the cross partial derivatives of the utility function W. To produce a definitive conclusion, one may

TABLE 1

SUMMARY OF THE COMPARATIVE STATICS RESULTS

Approach		Effect of an increase in the money growth rate on k	ℓ	k/ℓ	m	c	x	W
1. MIUF	I.	−	−	0	−	−	+	−
	II.	+	+	0	−	+	−	?
2. CIA	I.	−	−	0	−	−	+	−
	II.	−	?	−	−	−	?	−
3. TC		−	−	0	−	−	?	−

NOTES: MIUF I: $W_{c\ell} < 0$; W_{cm}, $W_{\ell m} > 0$ (Transactions service model)
MIUF II: $W_{c\ell} < 0$; W_{cm}, $W_{\ell m} < 0$ (Asset substitution model)
CIA I: $\Gamma = 0$ (Lucas model)
CIA II: $\Gamma > 0$ (Stockman model)

assume Pareto complementarity between consumption and leisure ($X = 1 - \ell$), Pareto complementarity between consumption and money, and Pareto substitutability between leisure and money, that is, $W_{c\ell} < 0$, $W_{cm} > 0$, and $W_{\ell m} > 0$, respectively. In this case, one may follow Fischer (1979) to restrict $dm/d\mu < 0$, and then obtain the reversed Tobin effect, that is, a high money growth rate leads to lower steady-state capital, labor, and consumption. The intuition is as follows. When $W_{cm} > 0$ and $W_{\ell m} > 0$, a lower money holding would reduce consumption but increase leisure; the former creates a lower derived demand for labor and the latter results in a lower labor effort under the time constraint. Thus steady-state labor declines and, by factor complementarity $f_{k\ell} > 0$ (a property of a constant-returns production technology), steady-state capital decreases as well. In this case, higher monetary growth is welfare reducing.

On the contrary, if $W_{c\ell} < 0$ but W_{cm}, $W_{\ell m} < 0$, then we obtain a conclusion analogous to the asset substitution model developed by Tobin (1965). Since we do not know theoretically what restrictions to impose on the cross-partial derivatives of the utility function, the comparative statics results turn out to be inconclusive overall. Nevertheless, the modified golden rule (3) is independent of the money growth rate. As a consequence of constant returns, the capital-labor ratio (k/ℓ) is unaffected by the rate of money growth.

2. THE CASH-IN-ADVANCE APPROACH

We next turn to develop a general cash-in-advance model in which all consumption goods and a fraction (Γ) of investment goods have to be purchased out of existing money balances:

$$m(t) \geq c(t) + \Gamma\dot{k}(t) , \tag{6}$$

where $0 \leq \Gamma \leq 1$. $\Gamma = 0$ captures the Lucas (1980) economy, while $\Gamma > 0$ characterizes the Stockman (1981) setup. Different from Lucas, we consider a production economy; in contrast to Stockman, we allow for endogenous labor-leisure choice. Given the instantaneous utility (U) as a function of consumption and leisure ($x = 1 - \ell$), the representative agent's optimization problem is given by

$$\max \int_0^\infty U(c(t), 1 - \ell(t))e^{-\rho t}dt \tag{7}$$

subject to the budget constraint (2) and the cash-in-advance (or finance or liquidity) constraint (6). Also, U is assumed to be well-behaved.

In the steady state, $\dot{c} = \dot{\ell} = \dot{m} = \dot{k} = 0$, $\pi = \mu - n$ and $c = f(k, \ell) - nk$, so the cash-in-advance constraint must be binding: $m = c = f - nk$.[2] As a consequence, one obtains

[2]When $m - c - \Gamma k > 0$, $\rho + \mu = 0$ and a knife-edged problem emerges.

$$f_k = \rho + n + \rho(\rho + \mu)\Gamma \quad (8)$$

$$f_\ell U_c = (1 + \rho + \mu)U_x . \quad (9)$$

Different from (3), the modified golden rule (8) depends *directly* on money growth, as does the no-arbitrage condition (9) as opposed to (4).

As Table 1 indicated, the effect of higher money growth is to decrease steady-state capital, real money balances, consumption, and welfare. Under Lucas' framework in which money is required only prior to purchasing consumption good (that is, $\Gamma = 0$), an increase in the rate of money growth raises money-holding costs and so decreases real balances and consumption under a binding cash-in-advance constraint. It, therefore, depresses the derived demand for production inputs, capital, and labor, and hence increases leisure (but the welfare gain from higher leisure is dominated by the welfare loss from lower consumption in the steady state). The steady-state capital-labor ratio is independent of all money matters in this case.

Alternatively, the Stockman-like finance technology (that is, $\Gamma > 0$) results in an ambiguous effect of money growth on steady-state labor/leisure choice. This is due to the production effect offsetting against the direct utility effect. Specifically, higher money growth reduces real balances and hence capital and consumption with a binding cash-in-advance constraint. Lower capital, under the complementarity of production inputs ($f_{k\ell} > 0$), depresses labor effort and, consequently, leisure increases (which is referred to as a production effect). Lower consumption, on the other hand, raises marginal utility of consumption and marginal cost of leisure. This then causes leisure to decrease (which is called a direct utility effect). Moreover, from (9), higher money growth increases steady-state marginal cost of capital and hence reduces the capital-labor ratio in the steady state. In contrast to Stockman (1981), the reversed Tobin effect obtains even when $\Gamma = 0$ in an economy with endogenous labor-leisure choice.

3. THE TRANSACTIONS-COSTS APPROACH

In this section, money is introduced into the optimization problem through a shopping-time technology analogous to that in Saving (1971): $T(t) = T(c(t), m(t))$, where T is a continuously twice-differentiable function, satisfying $T_c > 0$, $T_m < 0$, $T_{cc} > 0$, $T_{mm} > 0$, $T_{cm} \le 0$, $\lim_{m \to 0} T_m(c, m) = -\infty$, and $T(0, m) = 0$ for $c > 0$ and $m < q^*c < \infty$. When $m \ge q^*c$, money is sufficient to facilitate all transactions and transactions time cost is driven essentially down to zero, that is, $T = T_c = T_m = T_{cc} = T_{mm} = T_{cm} = 0$. Here $T_{cc} > 0$ implies that the transactions cost schedule is convex, while $T_{mm} > 0$ describes that the benefit of using money for transactions has diminishing returns. Under $T_{cm} \le 0$, higher real balances lead to unchanged or lower marginal transactions costs of consumption (see a discussion in Feenstra 1986). Moreover, the limit condition on T_m is used to ensure the existence of money. Notice that the shopping-time technology proposed by McCallum (1983) and Kimbrough (1986) can be regarded as a special case of ours in which T is assumed to be linearly homogenous, that is, $T(c, m) = s(m/c)c$. Also, it captures the Drazen (1979) shopping-time technology when T is homogeneous of degree zero, that is, $T = T(m/c)$. Further, the transactions cost setup here is different from that constructed by Dornbusch and Frenkel (1973) in which money enters the system through a "pecuniary" transactions cost function.[3]

The representative agent's instantaneous utility (U) depends on consumption and leisure ($x = 1 - \ell - T$). Thus the optimization problem becomes

$$\max \int_0^\infty U(c(t), 1 - \ell(t) - T(t))e^{-\rho t}dt \tag{10}$$

subject to the budget constraint (2) and the shopping-time technology specified above, for all $t \geq 0$. In contrast to Kimbrough (1986) who focused mainly on employment, capital accumulation is endogenously determined.

In the steady state, $\dot{c} = \dot{\ell} = \dot{m} = \dot{k} = 0$, $\pi = \mu - n$ and $c = f(k, \ell) - nk$. The steady-state fundamental equations can thus be derived as

$$f_k = \rho + n \tag{11}$$

$$f_\ell U_c = (1 + T_c f_\ell)U_x \tag{12}$$

$$-T_m f_\ell = \rho + \mu , \tag{13}$$

which equate marginal benefits of k, ℓ, and m with their marginal costs, respectively.

We present the comparative statics results in Table 1. We find that the effect of a higher money growth rate is to reduce steady-state capital, labor, real money balances, consumption, and welfare unambiguously. From the modified golden rule (11), the steady-state capital-labor ratio is independent of the rate of money growth under constant returns. The impact of the money growth rate on steady-state leisure in this case is generally ambiguous. Intuitively, an increase in the rate of money growth reduces real balances, consumption, and labor effort (that is, the reversed Tobin effect). The reduction in real balances increases shopping time. By normality, it then decreases leisure given a constant time endowment, which can be referred to as a real balance effect. On the other hand, the reduction in consumption causes shopping time to decrease, which leads to higher leisure given the lower labor effort. This can be called a consumption effect. In the case that the shopping-time function is not too convex and that the intertemporal elasticity of substitution for consumption is large enough around the steady state, the above mentioned consumption effect indeed dominates the real balance effect. As a consequence, the effect of money growth on steady-state leisure becomes positive. An extreme example is to specify a perfect intertemporal substitution for consumption (that is, U is

[3]With pecuniary transactions costs, the effects of money growth on all real variables are generally ambiguous; see Dornbusch and Frenkel (1973).

nearly linear around the steady state), steady-state leisure unambiguously increases with a higher money growth.

Finally, it is worth noting that when the shopping-time technology is linearly homogeneous, higher money growth results in a lower money-consumption ratio. Thus, in analogy to Kimbrough (1986), shopping time per unit of consumption good (s) increases unambiguously.

4. FURTHER DISCUSSION

We have examined the effects of the money growth rate on per capita real variables and economic welfare in the steady state under the money-in-the-utility function (MIUF), the cash-in-advance (CIA), and the transactions-costs (TC) approaches. In the MIUF model described in section 1, the conclusion is generally ambiguous, depending crucially on the cross-partial derivatives of the instantaneous utility function. This ambiguity is mostly removed once the exact role played by money is specified, as in the CIA and TC models.

We turn next to compare the model predictions under different approaches. The analysis is primarily motivated from the construction of functional equivalence by Feenstra (1986).[4] Using the functional equivalence analysis, Feenstra established an exact duality between a simple MIUF setup and a pecuniary liquidity-cost setup. This is done by transforming consumption and liquidity costs into a "gross consumption" variable so that both the objective function and the constraint(s) exhibit identical forms. In the presence of endogenous labor-leisure choice, it is impossible to obtain such an isomorphism by simply redefining the consumption variable. Specifically, if we follow Feenstra (1986) to generate identical budget constraints for MIUF and TC models, the different time constraints result in different identities for leisure and hence different objective functional forms. Although the redefinition of the consumption variable would not establish functional equivalence under endogenous labor-leisure setup, one can follow McCallum (1983) to create identical utility functional forms between the MIUF and the TC models. This is done by incorporating a "gross leisure" variable into the objective function of the MIUF framework, which is defined as the sum of leisure and shopping time.

In general, the application of the functional equivalence analysis relies essentially on redefining choice variables. It is, however, worth noting that when macroeconomists adopt the MIUF approach, they would not regard consumption or leisure as the one incorporating transactions cost terms. In other words, to obtain functional equivalence has altered the nature of the original construction of the traditionally used MIUF framework.

This motivates us to develop a "qualitative equivalence" between different models, which requires weaker conditions than the functional equivalence analysis. Specifically, we restrict our attention to conditions that lead to qualitatively equivalent

[4]In pecuniary transactions cost models functional equivalence requires complex utility functions that are not naturally used by economists.

comparative statics results under different approaches *with no changes in their model nature*. This development also enables us to see how robust a hypothesis is and whether a theoretical prediction depends crucially on the underlying model structure.

First, we establish an equivalence between the MIUF model and the CIA model in which money is only required prior to purchasing the consumption good (that is, $\Gamma = 0$). When the cash-in-advance constraint is binding, consumption equals real money balances. Thus, consumption and money are perfectly complementary in this case. We now assume that $W(c, \ell, m)$ satisfies $W_{c\ell} < 0$, $W_{cm} > 0$ and $W_{\ell m} > 0$ and then impose an ex post constraint $c = m$ in equilibrium, that is, consumption and money exhibit perfect Pareto complementarity. Then, from Table 1, MIUF(I) and CIA(I) frameworks lead to the same comparative statics results.

Next, we examine qualitative equivalence between the MIUF and the TC approaches. This can be done by setting $W(c, \ell, m) \equiv U(c, 1 - \ell - T(c, m))$. Thus, given $U_{cx} > 0$, straightforward derivations yield the following restrictions on cross-partial derivatives: $W_{c\ell} < 0$, $W_{cm} > 0$ and $W_{\ell m} > 0$. Intuitively, an increase in leisure or real balances increases the marginal valuation of consumption, while higher real balances save shopping time and thus decrease the marginal valuation of leisure. Assume direct and own effects dominate so that W is strictly increasing and strictly concave in c, $-\ell$, and m. Then, the comparative statics results under the MIUF framework are consistent with results in the TC model except for the prediction on leisure. When the real balance effect is dominated by the consumption effect (specifically, $\epsilon U_x/U_c \geq \xi_1 + \xi_2$, where $\epsilon = cT_c(T_{cm}/T_m - T_{cc}/T_c)$, $\xi_1 = -cU_{cc}/U_c$ and $\xi_2 = cU_{cx}/U_x$), the effect of higher money growth is to increase leisure in both models.[5]

We now study qualitative equivalence between the CIA and the TC approaches. Following McCallum (1983) and Kimbrough (1986), the CIA model with $\Gamma = 0$ is an extreme case of the TC model in which the shopping-time technology can be expressed as a step function described by

$$T(c,m) = \begin{cases} 1 & \text{if } m < c \\ 0 & \text{if } m \geq c\ . \end{cases} \tag{14}$$

Thus, under monotonicity of preferences and the Inada conditions, the optimal decision has to be associated with $m/c = q^* = 1$ and hence T and all its derivatives are zero. Assume that ϵ is strictly positive and finite. Further assume that the consumption effect of money growth on leisure weakly dominates the real balance effect, then all comparative-static results in the TC model are in analogy to those in the CIA model. Finally, when leisure is completely inelastic, both models lead to the superneutrality of money.

In Table 2, we summarize the conditions for qualitative equivalence between the

[5]When T is linearly homogeneous [as in the McCallum (1983) and Kimbrough (1986)], $\epsilon = -sqs_{qq}/s_q > 0$, where $q = m/c$.

TABLE 2

CONDITIONS FOR QUALITATIVE EQUIVALENCE

1. $W_{cx} > 0$ (Pareto complementarity between consumption and leisure)
2. $W_{cm} > 0$ (Pareto complementarity between consumption and money)
3. $W_{xm} < 0$ (Pareto substitutability between leisure and money)
4. $\epsilon U_x/U_c \geq \xi_1 + \xi_2$ (weakly dominant consumption effect of money growth compared to the real balance effect)

Results:

$$\frac{dk}{d\mu} < 0; \frac{d\ell}{d\mu} < 0; \frac{d(k/\ell)}{d\mu} = 0; \frac{dm}{d\mu} < 0; \frac{dc}{d\mu} < 0; \frac{dx}{d\mu} > 0; \frac{dW}{d\mu} < 0$$

three (MIUF, CIA, and TC) approaches in which all comparative statics results derived from these models are qualitatively identical.

Finally, we would like to caution the reader to recognize the limitation of the qualitative equivalence analysis. On one hand, there is no guarantee that the qualitative equivalence analysis would establish a unique set of restrictions. When comparative statics results for different models are generally ambiguous, one may find different sets of restrictions to obtain consistent predictions. On the other hand, there may exist no qualitative equivalence between some models. For example, if the cash-in-advance constraint is binding for both the consumption good and the investment good, there exists no qualitative equivalence between the cash-in-advance model and others. In particular, the inconsistent predictions of the effect of money growth on the steady-state capital-labor ratio become unavoidable.[6] It may be possible to study a generalized shopping-time function: $T = T(c, m, \dot{k})$, which is compatible with the generalized cash-in-advance setup with $\Gamma > 0$. In this case, however, capital enters the utility function and, as shown in Kurz (1968), multiple steady-state equilibria will emerge. As a consequence, we obtain inconclusive comparative static results and hence qualitative equivalence cannot be established. Another example is a simplified transactions-costs model in which shopping time is either independent of consumption or separable in consumption and real money balances. Under this specification, the effect of money growth on steady-state leisure can be shown to be unambiguously negative. Thus, one can never generate qualitative equivalence between the transactions-costs model and other approaches. Nevertheless, qualitative equivalence requires weaker conditions than functional equivalence. It is therefore impossible to obtain functional equivalence for any of the above-mentioned examples.

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