INTERNATIONAL ECONOMIC REVIEW
Vol. 38, No. 1, February 1997

ON THE EXISTENCE OF BALANCED GROWTH EQUILIBRIUM*

By THEODORE PALIVOS, PING WANG AND JIANBO ZHANG[1]

Louisiana State University, U.S.A., and Tilburg University, The Netherlands
The Pennsylvania State University, U.S.A.
The University of Kansas, U.S.A.

We characterize the class of dynamic models that allow for the most commonly used types of sustained economic growth (balanced and asymptotically balanced). We show that, under a constant returns to scale technology, (asymptotically) constant discount rate and (asymptotically) constant elasticity of marginal felicity are not only necessary but also sufficient conditions for the existence of $a(n)$ (asymptotically) balanced growth equilibrium path. We provide examples of recursive utility models that accept $a(n)$ (asymptotically) balanced growth equilibrium and discuss their implications on cross-country differences in growth rates, as well as on savings behavior and wealth distribution.

1. INTRODUCTION

This paper examines the possibility of sustained optimal growth in a continuous-time model where the representative agent has a variable elasticity of marginal felicity and/or a variable rate of subjective discount rate. It establishes necessary and sufficient conditions for the existence of balanced growth and asymptotically balanced growth paths. Under a linear technology, these necessary and sufficient conditions involve restrictions on the structure of intertemporal preferences.

Recently, there has been an increasing interest in uncovering the determinants of the economic growth rate (e.g., Romer 1986, Lucas 1988, and Rebelo 1991). Most of the literature, however, has been based on models with a stylized intertemporal preference structure, taking the intertemporal elasticity of substitution in consumption and the rate of time preference as constant.[2] Accordingly, cross-country differences in growth rates are exclusively attributed to differences in returns to investment.[3] The present study extends the endogenous growth literature by allowing for a general recursive intertemporal preference structure, which can further

* Manuscript received July 1993; revised December 1994.

[1] We are grateful to John Boyd III, Wilfred Ethier, Lionel McKenzie, Henry Wan and two anonymous referees of this journal for helpful comments and suggestions. We have also benefited from comments by seminar participants at Academia Sinica, Taiwan, the University of Rochester, and the Fall 1994 Midwest Mathematical Economics Meetings, U.S.A.

[2] An exception is the model presented in Rebelo (1992), which attributes differences in the growth rates to differences in the (endogenous) elasticity of intertemporal substitution in consumption.

[3] Should the constant preference parameters differ across countries, perfect capital mobility implies that only the least impatient country will hold capital and that the perfect foresight steady-state equilibrium is degenerate for all other countries (see Becker 1980).

promote our ability to explain cross-country differences in growth patterns,[4] as well as savings behavior and wealth distribution.[5]

We utilize the preference setup introduced by Uzawa (1968) and recently extended and clarified by Epstein (1987), Becker, Boyd, and Sung (1989), and Obstfeld (1990).[6] This is the most general recursive utility functional, defined in continuous time, that has been used in the literature to analyze issues regarding intertemporal behavior with endogenous discounting.[7] Within this recursive utility framework and under a linear technology, we first examine the possibility of balanced growth. We show that constant (over time) elasticity of marginal felicity and discount rate are not only necessary but also sufficient conditions for the existence of a balanced growth equilibrium path. We then extend our analysis to the case of asymptotically balanced growth paths, defined as solutions to an optimal growth problem such that all variables grow at asymptotically constant growth rates. Under some regularity conditions, which ensure well-behaved preferences and positive discounting, we show that asymptotically constant elasticity of marginal felicity and discount rate are necessary and sufficient for the existence of a unique, nondegenerate, asymptotically balanced growth path.[8] Thus, our paper completely characterizes the class of dynamic models which allow for the most commonly used types of sustained economic growth (balanced and asymptotically balanced). Finally, to illustrate our main results, we provide several examples/economies, which accept either a balanced growth or an asymptotically balanced growth path, and analyze the endogenously determined rate of economic growth.

The remainder of the paper is organized as follows. In Section 2, we describe the model/economy with variable discount rate and obtain the first-order conditions for the optimal growth problem. The model presented in that section is one of the most general among the class of optimal growth models with one capital stock, in the sense that it considers general felicity and discounting functions. Section 3 provides necessary and sufficient conditions for the existence of a balanced-growth equilibrium, while Section 4 establishes parallel conditions for the existence of an

[4] The need to understand the determinants of the (subjective) discount rate has been pointed out in the literature (Sala-i-Martin 1990, p. 2), and the consideration of the endogenous growth process in the case of an endogenous rate of time preference has already been suggested (Obstfeld 1990, p. 72).

[5] Within the exogenous growth framework with recursive preferences, rich implications on savings behavior and wealth distribution can be drawn from the analyses in Epstein and Hynes (1983), and Epstein (1987).

[6] The Uzawa–Epstein framework has been widely applied in areas where the implications of a constant rate-of-time preference are particularly unappealing (e.g., Obstfeld 1982, Epstein and Hynes 1983, and Devereux 1991).

[7] Another class of recursive preferences, defined in continuous time, is the habit-formation framework, as studied, for example, by Ryder and Heal (1973), and Becker and Murphy (1988). Additionally, Shi and Epstein (1993) propose a utility functional that incorporates both habit formation and endogenous rate-of-time preference.

[8] The relationship between the elasticity of marginal felicity and the intertemporal elasticity of substitution, as well as between the discount rate and the rate-of-time preference, is analyzed below. At this point it should be noted that, unlike the time-additive framework, in the case of recursive preferences these concepts are quite different.

asymptotically balanced-growth equilibrium. Finally, Section 5 provides examples to illustrate the main results and Section 6 concludes the paper.

2. THE MODEL

Consider an economy with constant population, zero depreciation rate and linear technology, in which a representative agent seeks to maximize her lifetime discounted utility with variable discount rate:

$$(P) \qquad \max \Omega(C) = \int_0^\infty u(c) e^{-\Delta(t)}\, dt,$$

subject to

$$(1a) \qquad \dot{k} = Ak - c,$$

$$(1b) \qquad \dot{\Delta} = \theta(c),$$

$$(1c) \qquad k(0) = k_0 > 0.$$

Following conventional terminology, we refer to $u(\cdot)$ as the *felicity*, to $\theta(\cdot)$ as the *(subjective) discount rate*, and to $\Delta(\cdot)$ as the *cumulated (subjective) discount rate*. c, $A(0 < A < \infty)$, and k, on the other hand, denote, respectively, per capita consumption, a production-scaling factor, and a composite of per capita physical and human capital stocks. The set of admissible consumption paths consists of paths $C = \{c(t) | c(t) > 0, \forall t \geq 0\}$.[9] Furthermore, throughout the paper, $\dot{x}$ and $\ddot{x}$ denote the first and second time derivative of any variable x. Finally, as in any growth model, the capital accumulation is governed by nonconsumed output (equation 1a).

The preference structure, as specified in (P), is *recursive* in that $\Delta(\cdot)$ is allowed to depend on the agent's past and current consumption levels, as described by equation (1b). The particular functional adopted here is Epstein's generalization of the Uzawa (1968) functional (see Epstein 1987).[10] We also require:

ASSUMPTION 1. *The felicity function* u: $\mathbb{R}_+ \to \mathbb{R}$ *is twice continuously differentiable, with* $u'(c) > 0$ *and* $u''(c) \leq 0$, $\forall c > 0$.

ASSUMPTION 2. *The (subjective) discount rate* θ: $\mathbb{R}_+ \to \mathbb{R}_{++}$ *is twice continuously differentiable and satisfies the following properties*:

$$(i)\ \theta'(c) > 0, \text{ and } \theta''(c) < 0\ \forall c > 0 \text{ and } (ii)\ \Delta(0) = 0.$$

Assumptions 1 and 2 are very common in the literature, although there is considerable disagreement over whether $\theta'(c)$ is positive or negative (for a survey, see

[9] In general, we can allow c to be a vector and $\theta(\cdot)$ to depend on a transformed consumption variable. For illustrative convenience, however, we present our results using a single consumption good. In Section 5 below, we provide examples that either allow for more than one argument in the felicity function or modify the discounting function to depend on the consumption–capital ratio.

[10] Uzawa's specification can easily be recast into Epstein's recursive utility framework under proper conditions (see Nairay 1984, and Obstfeld 1990).

Epstein 1987, and Obstfeld 1990). Note, however, that $\theta'(c) > 0$ is needed for convergence and hence imposed hereafter.

Under this recursive utility specification, one needs to define marginal utility and the rate-of-time preference with caution. Following Epstein (1987), let ${}_TC$ denote the tail of C at time T, which generates a utility level of $\Omega({}_TC) = \int_T^{\infty} u(c)\exp[-\int_T^t \theta(c)\,ds]\,dt$. The Volterra derivative of Ω with respect to $c(t)$, $\Omega_T(C)$, which measures the increment in lifetime utility caused by a small increase in consumption along path C and at times near T, can be used to define a generalized notion of the marginal utility of consumption at time T:[11]

$$\Omega_T(C) = \left[u'(c(T)) - \theta'(c(T))\Omega({}_TC)\right]\exp\left[-\int_0^T \theta(c)\,ds\right].$$

We next compute the rate of time preference, ρ, defined as the rate at which the marginal utility of consumption falls along a *locally constant* path:

$$\rho(c, \Omega({}_TC)) \equiv -\frac{d}{dT}\log \Omega_T(C)\big|_{\dot{c}(t)=0} = \theta(c)\frac{u'(c) - [u(c)/\theta(c)]\theta'(c)}{u'(c) - \theta'(c)\Omega({}_TC)}$$

One can easily check that in the case of a time-additive preference structure, $\theta'(c) = 0$ and thus $\rho = \theta$. In the general recursive utility framework, however, the rate of time preference at any point in time depends on both the consumption level at time T, $c(T)$, and the utility of the tail of the consumption path, $\Omega({}_TC)$.

ASSUMPTION 3. (*i*) $u'(c(T)) - \theta'(c(T))\Omega({}_TC) > 0 \ \forall T \geq 0$. (*ii*) $u'(c) > [u(c)/\theta(c)]\theta'(c) \ \forall c > 0$. (*iii*) $u''(c) < [u'(c)/\theta'(c)]\theta''(c)$, $\forall c > 0$.

Assumptions (3*i*) and (3*ii*) ensure that the marginal utility and the rate-of-time preference are positive, while Assumption (3*iii*) implies that the Hamiltonian (specified below) is jointly strictly concave in c and k. These conditions are the familiar ones for monotonicity and concavity in Epstein (1987).

As documented in Becker, Boyd and Sung (1989, Section 4.2), in the case where the discount rate is positive [$\theta(c) > 0$], nonincreasing returns-to-scale technology is required for the existence of an optimal path. Furthermore, if $\theta' \geq 0$, then it is straightforward to show that nondecreasing returns-to-scale technology is required for the existence of perpetual growth (see Romer 1986, and Jones and Manuelli 1990 for the case of an economy with time-additive preference structure). We, therefore, consider the linear technology (Ak) introduced by Gale and Sutherland (1968) and also considered in Rebelo (1991). Two points concerning this linear technology are noteworthy. First, the marginal product of capital (MPK) is constant with respect to k. As suggested by Jones and Manuelli (1990), however, it suffices to assume that the marginal product of capital is bounded away from zero. An example of such a production function is the generalized Gale–Sutherland function, $Ak + Bk^b$, where $1 > b > 0$ and $\infty > A$, $B > 0$. Although using this production function

[11] See Wan (1970), Ryder and Heal (1973) and Epstein (1987) for further details.

instead of the "Ak" will not change qualitatively the asymptotic behavior of the system, it will complicate the analysis significantly. Second, the MPK is constant over time. This attribute is present in most of the endogenous growth models in which a balanced growth path exists (e.g., Lucas 1988, Barro 1990, and Rebelo 1991), and is one of the stylized facts of growth as described by Kaldor (1961).

We further impose the following assumption:

ASSUMPTION 4. *$u(c)/\theta(c) \leq Qe^{\beta\Delta(c)} \forall c > 0$, where $0 < Q < \infty$ and $\beta < 1$.*

Assumption 4 ensures that the lifetime utility (Ω) in (P) is bounded and hence the optimization problem is well-defined. To see this, we note that

$$\Omega(C) = \int_0^\infty u(c)e^{-\Delta(t)}\,dt = \int_0^\infty \frac{u(c)}{\theta(c)}e^{-\Delta}\,d\Delta \leq \int_0^\infty Qe^{-(1-\beta)\Delta}\,d\Delta = \frac{Q}{1-\beta}.$$

To perform the maximization suggested above (see program (P)) consider the following Hamiltonian:

$$H(c, k, \Delta, \tilde{\lambda}, \tilde{\mu}) = u(c)e^{-\Delta} + \tilde{\lambda}[Ak - c] + \tilde{\mu}\theta(c),$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ denote the costate variables associated with (1a) and (1b), respectively. Applying the Pontryagin maximum principle, we get the following first-order necessary conditions:

$$u'(c)e^{-\Delta} + \tilde{\mu}\theta'(c) = \tilde{\lambda}, \tag{1d}$$

$$\dot{\tilde{\lambda}} = -A\tilde{\lambda}, \tag{1e}$$

$$\dot{\tilde{\mu}} = u(c)e^{-\Delta} \tag{1f}$$

together with (1a), (1b) and the transversality condition[12]

$$\lim_{t\to\infty} H(t) = 0. \tag{1g}$$

Next we reduce the system consisting of equations (1a)–(1g) into a more tractable one that involves only one state variable, $k(t)$. First define $\lambda(t) \equiv \tilde{\lambda}(t)e^{\Delta(t)}$, and $\mu(t) \equiv \tilde{\mu}(t)e^{\Delta(t)}$. Next recall that, along the optimal trajectory, $dH/dt = \partial H/\partial t$ (see Intriligator 1971, p. 350). Since the specific problem considered here is autonomous, $\partial H/\partial t = 0$, and thus the Hamiltonian is independent of time along the optimal trajectory. This and the transversality condition (1g) imply that the Hamiltonian

[12] On the transversality condition in infinite horizon, continuous-time optimization problems, see Michel (1982).

takes the value zero along the optimal trajectory. Thus,[13]

$$\mu = -\frac{1}{\theta(c)}\left[u(c) + \lambda(Ak - c)\right].$$

Substituting this relationship, (1a), and the definitions of λ and μ in (1d) and (1e) yields

(2a) $$\frac{\theta'(c)}{\theta(c)}\{u(c) + \lambda\dot{k}\} - u'(c) + \lambda = 0,$$

(2b) $$\frac{\dot{\lambda}}{\lambda} = \theta(c) - A.$$

In the following sections, we analyze the dynamic system consisting of (1a), (1c), (2a), and (2b) and derive conditions for the existence of balanced and asymptotically balanced growth.[14]

3. BALANCED GROWTH

We first define the concept of *balanced growth.*[15]

DEFINITION 1. Let a path $\{k(t), c(t), \lambda(t)\}$, $t \geq 0$, be a solution to (P). We call it a *balanced growth (equilibrium) path* if the growth rates of all these variables, $\eta_k = \dot{k}/k$, $\eta_c = \dot{c}/c$ and $\eta_\lambda = \dot{\lambda}/\lambda$, are constant over time. A balanced growth path is said to be *nondegenerate* if η_k and η_c are strictly positive.

A degenerate balanced growth path corresponds to a (nongrowing) stationary state in the usual sense. Since we are interested in economies that exhibit perpetual growth, we often refer to a nondegenerate balanced growth path simply as a balanced growth path without special notification. Furthermore, we assume:

ASSUMPTION 5. *The subjective discount rate is less than the marginal product of capital, that is,* $\theta(c) < A$, $\forall c \geq 0$.

This is also a very common assumption in the literature of endogenous growth; it ensures a positive growth rate of consumption (for example, Lucas 1988, Barro 1990, Jones and Manuelli 1990, and Rebelo 1991).

[13] Alternatively, one can use the Hamilton–Jacobi equation for this problem: $0 = -(\partial J/\partial t) = \max_c u(c)e^{-\Delta} + \tilde{\lambda}[Ak - c] + \tilde{\mu}\theta(c)$, where J denotes the value function of the problem and it is assumed to be differentiable.

[14] We do so following an indirect approach, based on Pontryagin's necessary conditions for optimality, rather than a direct method as in Becker and Boyd (1992).

[15] The term *balanced growth* was originally used by von Neumann to describe an economy with a common growth rate. The more general concept, followed here, has been used in the dynamic Leontief model of Dorfman, Samuelson and Solow (1958), and in several recent endogenous growth papers, e.g., Lucas (1988), and Rebelo (1991).

We next present the necessary conditions for the existence of a balanced growth path. The following proposition indicates that in order for an economy to exhibit balanced growth it is required that both the discount rate, $\theta(c)$, and the elasticity of marginal felicity, $\sigma(c) \equiv -u''(c)c/u'(c)$, be constant.[16]

PROPOSITION 1. *Under Assumptions 1–5, if an economy accepts a balanced growth equilibrium path, then along such a path the functionals $\theta(c)$ and $\sigma(c)$ are constant over time.*

PROOF. Equation (2b) implies

$$\eta_\lambda = \dot{\lambda}/\lambda = \theta(c) - A.$$

Hence, along the balanced growth path, $\theta(c) = \eta_\lambda + A$ is constant. In this case, equation (2a) reduces to the usual condition

$$\lambda = u'(c).$$

Thus,

$$\eta_\lambda = \frac{u''(c)\dot{c}}{u'(c)} = \eta_c \frac{u''(c)c}{u'(c)},$$

implying,

$$\sigma(c) \equiv -\frac{u''(c)c}{u'(c)} = -\frac{\eta_\lambda}{\eta_c} = \text{constant}. \qquad \text{Q.E.D.}$$

Next, we show that the converse of Proposition 1 is also true.

PROPOSITION 2. *Under Assumptions 1–5, if there exists a path along which the discount rate and the elasticity of marginal felicity are constant over time, then it is a balanced growth path.*

PROOF. Under Assumptions 1–5, constant discount rate and constant intertemporal elasticity of substitution, it is easily verified that the system, (1a), (2a), and (2b), is equivalent to

$$\frac{\eta_\lambda}{\eta_c} = -\sigma,$$

$$\eta_\lambda = \theta - A,$$

$$\eta_k = A - \frac{c}{k},$$

[16] Under a time-additive preference structure the inverse of σ is equal to the elasticity of intertemporal substitution in consumption. In the framework used here, however, this is not the case because the discount rate depends also on c. One can, instead, define the intertemporal elasticity of substitution in terms of psychological time, Δ, using the "transformed" felicity function $V(c) \equiv u(c)/\theta(c)$, as $-V'(c)/V''(c)c$.

or

$$\eta_c = \eta_k = \frac{A - \theta}{\sigma}; \qquad \eta_\lambda = \theta - A,$$

which constitutes a balanced growth path. Q.E.D.

In summary, when the discount rate is bounded from above by the marginal product of capital, constant (over time), along a particular path, discount rate and elasticity of marginal felicity are both necessary and sufficient conditions for the existence of a balanced growth equilibrium path. For a complete characterization of such a path in the "Ak" model, the reader is referred to Rebelo (1991).

4. ASYMPTOTICALLY BALANCED GROWTH

This section establishes necessary and sufficient conditions for the existence of an asymptotically balanced growth equilibrium path. We begin with the following definition.

DEFINITION 2. Let a path $\{k(t), c(t), \lambda(t)\}$, $t \geq 0$, be a solution to (P). It is said to be an *asymptotically balanced growth (equilibrium) path* if $\eta_k = \lim_{t\to\infty}(\dot{k}/k)$, $\eta_c = \lim_{t\to\infty}(\dot{c}/c)$, and $\eta_\lambda = \lim_{t\to\infty}(\dot{\lambda}/\lambda)$ exist and are finite. Furthermore, we say that an asymptotically balanced growth path is *nondegenerate* if η_k and η_c are strictly positive a.e.

Unless otherwise specified, we again restrict our attention to nondegenerate, asymptotically balanced growth equilibrium paths and we often refer to them simply as asymptotically balanced growth paths. For technical convenience, we make one additional assumption.

ASSUMPTION 6.

$$-\frac{\theta''(c)c}{\theta'(c)} \leq M \textit{ for some } M \in \mathbb{R}_+.$$

This assumption requires the elasticity of marginal discounting, $\theta'(c)$, not to be infinitely sensitive with respect to changes in consumption.

4.1. *Necessity.* The following lemmas, which hold under Assumptions 1–6, are used to derive necessary conditions for the existence of an asymptotically balanced growth path.

LEMMA 1. *If* $\lim_{c\to\infty}\theta(c) = \theta$ *is finite, then* $\lim_{c\to\infty}\theta'(c)c = 0$.

PROOF. Since $\theta'(c)c$ is positive, it is sufficient to show that $\lim_{c\to\infty}\sup\theta'(c)c = 0$. Suppose not. Then there exists $\delta > 0$ and a sequence $\{c_n\}\uparrow\infty$ such that

$$\theta'(c_n)c_n \geq \delta. \tag{3}$$

Without loss of generality, we can pick $\{c_n\}$ in such a way that

$$c_n > 2c_{n-1},\; n = 1,2,\ldots. \tag{4}$$

Combining (3) and (4), we have

$$\theta'(c_n)(c_n - c_{n-1}) \geq \delta\left[1 - \frac{c_{n-1}}{c_n}\right] > \delta/2.$$

By the Mean Value Theorem, we know there exists $\xi_n \in (c_{n-1}, c_n)$ such that

$$\theta(c_n) - \theta(c_{n-1}) = \theta'(\xi_n)(c_n - c_{n-1}).$$

Since $\theta'' < 0$, we have $\theta'(\xi_n) > \theta'(c_n)$ and thus

$$\theta(c_n) - \theta(c_{n-1}) = \theta'(\xi_n)(c_n - c_{n-1}) > \delta/2,$$

or by summing up both sides for $n = 1,2,\ldots,$

$$\lim_{n\to\infty} \theta(c_n) = \infty,$$

which contradicts the assumption that $\lim_{c\to\infty}\theta(c) = \infty$ is finite. Q.E.D.

LEMMA 2. *If* $\lim_{c\to\infty}\theta(c) = \theta$ *is finite, then* $\lim_{c\to\infty}\theta''(c)c^2 = 0$.

PROOF. From Assumption 6, we have

$$\lim_{c\to\infty} |\theta''(c)c^2| = \lim_{c\to\infty} |\theta''(c)c|c \leq \lim_{c\to\infty} M\theta'(c)c,$$

which is zero from Lemma 1. Q.E.D.

LEMMA 3. *If the economy moves along a nondegenerate, asymptotically balanced growth path then* c/k *is asymptotically constant.*

PROOF. From equation (1a) we know that

$$\frac{\dot{k}}{k} = A - \frac{c}{k}.$$

Thus, c/k converges asymptotically to $A - \eta_k$, which is constant. Q.E.D.

Utilizing Lemmas 1–3, we are now prepared to prove the following proposition, which establishes necessary conditions for the existence of an asymptotically balanced growth path. It shows that for such a path to exist, agents must have an asymptotically constant discount rate as well as asymptotically constant elasticity of marginal felicity.

PROPOSITION 3. *Under Assumptions* 1–6, *if an economy allows for a nondegenerate, asymptotically balanced growth path, then both* $\lim_{c\to\infty}\theta(c)$ *and* $\lim_{c\to\infty}\sigma(c)$ *exist and are finite.*

PROOF. Since $\dot{\lambda}/\lambda \to \eta_\lambda$, from equation (2b), we have

$$\eta_\lambda = \lim_{t\to\infty} \theta[c(t)] - A.$$

From Definition 2, an asymptotically balanced growth path implies that $c(t) \to \infty$ as $t \to \infty$. Hence, it follows that

$$\lim_{c\to\infty} \theta(c) = \eta_\lambda + A,$$

which is finite.

We next use equation (2a) to solve for λ,

$$\lambda = \frac{\theta(c)u'(c) - \theta'(c)u(c)}{\theta(c) + \theta'(c)(Ak - c)}. \tag{5}$$

Therefore,

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{c}[\theta(c)u''(c) - \theta''(c)u(c)]}{\theta(c)u'(c) - \theta'(c)u(c)} - \frac{\dot{c}[\theta'(c) + \theta''(c)(Ak - c)] + \theta'(c)(A\dot{k} - \dot{c})}{\theta(c) + \theta'(c)(Ak - c)}. \tag{6}$$

As $t \to \infty$, the left-hand side of (6) approaches η_λ. To show that the agent has asymptotically constant elasticity of marginal felicity, we need to verify that the right-hand side of (6) converges to a constant multiplied by the limit of $\sigma(c) \equiv -u''(c)c/u'(c)$. First, notice that the first term on the right-hand side of (6) is equivalent to

$$\frac{\dot{c}}{c}\,\frac{\dfrac{u''(c)c}{u'(c)} - \dfrac{\theta''(c)c^2}{\theta(c)}\dfrac{u(c)}{u'(c)c}}{1 - \dfrac{\theta'(c)c}{\theta(c)}\dfrac{u(c)}{u'(c)c}},$$

which, by Lemmas 1 and 2, converges to

$$-\eta_c \lim_{c\to\infty} \sigma(c).$$

Second, from Lemma 3, the last term on the right-hand side of (6) is equivalent to

$$\frac{(\dot{c}/c)\left[\theta'(c)c + \theta''(c)c^2(Ak/c - 1)\right] + \theta'(c)c\left[A(\dot{k}/k)(k/c) - (\dot{c}/c)\right]}{\theta(c) + \theta'(c)c[A(k/c) - 1)]},$$

which converges to zero by Lemmas 1, 2, and 3. Hence, the entire right-hand side of (6) converges to $-\eta_c \lim_{c\to\infty}\sigma(c)$ and thus

$$\lim_{c\to\infty} \sigma(c) = -\frac{\eta_\lambda}{\eta_c}, \tag{7}$$

which is finite. Q.E.D.

4.2. *Sufficiency.* The following lemma examines the asymptotic properties of the investment–consumption ratio, which are used below to establish sufficient conditions for a unique, asymptotically balanced growth path.

LEMMA 4. *If the asymptotic value of k/c is finite, that is, $\lim_{t\to\infty} k/c < \infty$, then the asymptotic values of $\dot{k}/c$ and $\ddot{k}/c$ are also finite, that is, $\lim_{t\to\infty}\dot{k}/c < \infty$ and $\lim_{t\to\infty}\ddot{k}/c < \infty$.*

PROOF. The resource constraint, (1a), immediately implies

$$\frac{\dot{k}}{c} = A\frac{k}{c} - 1.$$

Also, differentiating (1a) and then dividing both sides by c, we have

$$\frac{\ddot{k}}{c} = A\frac{\dot{k}}{c} - \frac{\dot{c}}{c}.$$

From the last two equations, it follows that $\lim_{t\to\infty} k/c < \infty$ implies $\lim_{t\to\infty}\dot{k}/c < \infty$ and $\lim_{t\to\infty}\ddot{k}/c < \infty$. Q.E.D.

We are now ready to establish the main result of the paper by showing that the converse of Proposition 3 is also true; namely, if the agents' discount rate and elasticity of marginal felicity are asymptotically constant then the economy follows an asymptotically balanced growth path.

PROPOSITION 4. Main Result. *Under Assumptions 1–6, if $\lim_{c\to\infty}\sigma(c) = \sigma < \infty$ and $\lim_{c\to\infty}\theta(c) = \theta < A$, then there exists a unique, nondegenerate, asymptotically balanced growth path.*

PROOF. By eliminating λ from the system (1a), (2b) and (5), we obtain the following dynamic system:

$$\theta(c) - A = \frac{\dot{c}[\theta(c)u''(c) - \theta''(c)u(c)]}{\theta(c)u'(c) - \theta'(c)u(c)} - \frac{\dot{c}[\theta'(c) + \theta''(c)(Ak - c)] + \theta'(c)(A\dot{k} - \dot{c})}{\theta(c) + \theta'(c)(Ak - c)}, \tag{8}$$

and (1a) ($\dot{k} = Ak - c$).

To show that a unique, asymptotically balanced growth equilibrium path exists, it is sufficient to show that the system has a unique solution $\{(k(t), c(t))\}$, $t \geq 0$, along which both consumption and capital stock grow at an asymptotically constant, positive rate. We undertake this task in two steps. First, we show the system to have a unique path such that both $c(t)$ and $k(t)$ approach infinity as $t \to \infty$ while $\lim_{t \to \infty}\{c(t)/k(t)\}$ exists and is finite. Second, we prove that along this path the growth rates of $c(t)$ and $k(t)$ converge to a positive constant as $t \to \infty$.

Step 1. The Existence of a Unique, Unbounded, Optimal Growth Path. Consider first the loci $\dot{c} = 0$ and $\dot{k} = 0$, given by equations (8) and (1a), respectively. They can be written as

$$A = \theta(c) + \theta'(c)[Ak - c], \tag{9a}$$

$$c = Ak. \tag{9b}$$

It can readily be shown that both curves are upward-sloping [recall that $\theta''(c) < 0$]. Moreover, $\dot{c}$ is positive (negative) above (below) the curve $\dot{c} = 0$, and $\dot{k}$ is positive (negative) below (above) the curve $\dot{k} = 0$. To verify the former, one can show that $\dot{c} > 0$ as $\dot{k} = 0$ and $c \to \infty$. To verify the latter, notice that $\dot{k} > 0$ if $k > 0$ and $c = 0$. It can also be easily verified that both c and k must go to infinity along the two paths given by $\dot{c} = 0$ and $\dot{k} = 0$. Furthermore, notice that the $\dot{c} = 0$ locus must lie below $\dot{k} = 0$, since $\theta(c) < A$ and $\theta'(c) > 0$. The two loci are depicted in Figure 1. Next we label various points and curves in Figure 1. First, let $c = \zeta(k)$ denote the locus $\dot{c} = 0$. Second, pick any point (k_1, c_1) on $c = \zeta(k)$, such that the slope of the tangent, $c = \omega(k)$, at that point is less than A, i.e., $c = \omega(k)$ solves $c = c_1 + \zeta'(k_1)(k - k_1)$, where $0 < \zeta'(k_1) < A$. In Figure 1, (k_1, c_1) is denoted as B_1. Third, define

$$S_k \equiv \{k | k \geq k_1, \zeta(k) = \omega(k)\} \text{ and } k_2 \equiv \sup_{k \in S_k} k,$$

that is, k_2 is the last point of intersection of $\zeta(k)$ and $\omega(k)$. Notice that S_k and k_2 are well-defined since $\lim_{k \to \infty}\zeta'(k) = 0$. We also denote the point (k_2, c_2), where $c_2 = \zeta(k_2)$, as B_2. Finally, consider the function

$$c = \phi(k) \equiv \Gamma_k \zeta(k) + (1 - \Gamma_k)\omega(k),$$

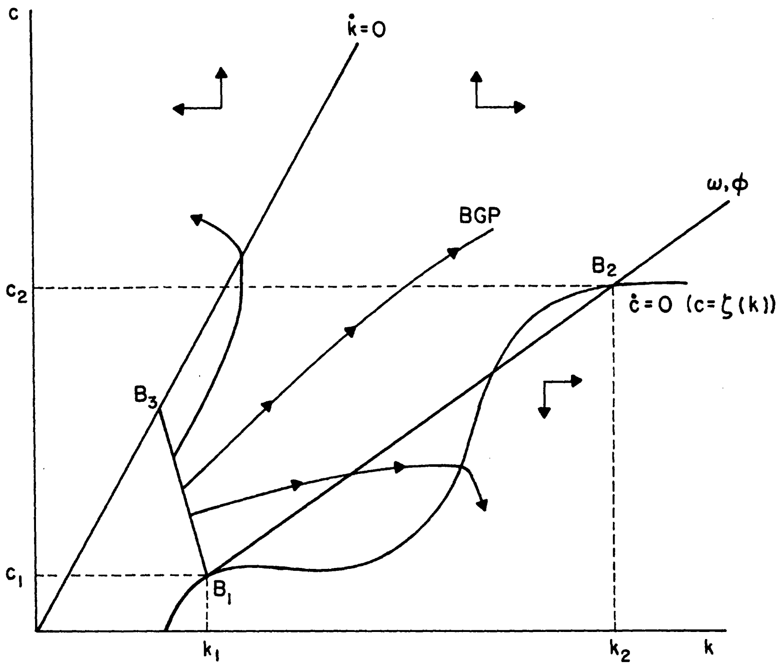

FIGURE 1

ASYMPTOTICALLY BALANCED GROWTH EQUILIBRIUM PATH

where

$$\Gamma_k = \begin{cases} 1 & \textit{if} \quad k < k_2 \\ 0 & \textit{if} \quad k \geq k_2 \end{cases},$$

is an indicator function. Put simply, $\phi(k)$ coincides with $\zeta(k)$ between points B_1 and B_2 and with $\omega(k)$ beyond B_2. Since $\phi(k)$ lies below $\dot{k} = 0$ and coincides or lies above $\dot{c} = 0$, between $\dot{k} = 0$ and $\phi(k)$ we have $\dot{k} > 0$ and $\dot{c} > 0$. Next pick any point B_3 on $\dot{k} = 0$ and consider solutions of (9) with initial condition on $\overline{B_1 B_3}$. If the initial condition is close to B_1 (B_3) then the path will cross $c = \phi(k)$ ($\dot{k} = 0$). Thus, by continuity, there must be a path starting from $\overline{B_1 B_3}$ that crosses neither $c = \phi(k)$ nor $\dot{k} = 0$ (indicated as BGP in Figure 1). Since this path has a positive slope and is bounded by $c = \phi(k)$ and $\dot{k} = 0$, it follows that both $c(t)$ and $k(t)$ grow unboundedly but $c(t)/k(t)$ is bounded,

$$0 < \lim_{t \to \infty} \inf \frac{c(t)}{k(t)} \leq \lim_{t \to \infty} \sup \frac{c(t)}{k(t)} = A < \infty,$$

along this path. It follows from the Poincaré–Bendixson theorem (see Boyce and DiPrima 1977, Theorem 9.9, p. 446) that as $t \to \infty$, $c(t)/k(t)$ either fluctuates within a closed interval (limit cycle) or converges to a constant value. Suppose that as $t \to \infty$, $c(t)/k(t)$ fluctuates within a closed interval. Then $\log[c(t)/k(t)]$ and $d\log[c(t)/k(t)]/dc(t)$ must also fluctuate as $t \to \infty$. Simple differentiation, however, in conjunction with (1a), yields $d\log[c(t)/k(t)]/dc(t) = [1/c(t)] - [dk(t)/dc(t)]/k(t) = 1/c(t)\{1 - [A - c(t)/k(t)]/(\dot{c}(t)/c(t))\}$. As $t \to \infty$, unbounded nondegenerate growth implies that $\dot{c}(t)/c(t)$ is bounded below from zero while $1/c(t)$ approaches to zero. As a consequence, $\lim_{t\to\infty}\{d\log[c(t)/k(t)]/dc(t)\} = 0$, which is a contradiction. Therefore, $c(t)/k(t)$ converges to a constant value and hence $\lim_{t\to\infty}[c(t)/k(t)]$ exists.

It remains to be shown that such an unbounded optimal growth path is unique. Suppose that there exist two optimal paths, which start from two distinct points on $\overline{B_1B_3}$, and both lead to unbounded growth. Note, also, that the configuration of the map must be such that these two paths cannot cross each other. Hence, the path that starts closer to B_3 must have a higher consumption level than the other $\forall t \geq 0$. But then, by strong monotonicity of preferences (see Assumption 3), the former must be strictly preferred to the latter, which contradicts the fact that both paths are optimal. This proves the uniqueness of the optimal path.

Step 2. The Path Found in Step 1 is Nondegenerate and Asymptotically Balanced. Equation (8) can be written as

$$\theta(c) - A = -\frac{\dot{c}}{c}\,\frac{\sigma(c) + \dfrac{\theta''(c)c^2}{\theta(c)}\dfrac{u(c)}{u'(c)c}}{1 - \dfrac{\theta'(c)c}{\theta(c)}\dfrac{u(c)}{u'(c)c}}$$

$$-\frac{(\dot{c}/c)\left[\theta'(c)c + \theta''(c)c^2(\dot{k}/c)\right] + \theta'(c)c(\ddot{k}/c)}{\theta(c) + \theta'(c)c(\dot{k}/c)}.$$

Let $t \to \infty$, hence $c \to \infty$. Using Lemmas 1, 2, and 4, the equation above is reduced to

$$\theta - A = \lim_{t\to\infty}\frac{\dot{c}}{c}(-\sigma),$$

and hence

$$\lim_{t\to\infty}\frac{\dot{c}}{c} = \frac{A-\theta}{\sigma}. \tag{10}$$

Next we characterize the asymptotic growth rate of $k(t)$. Applying L'Hôpital's rule, we obtain

$$\lim_{t\to\infty}\frac{c(t)}{k(t)} = \lim_{t\to\infty}\frac{\dot{c}(t)}{\dot{k}(t)}.$$

Using this relationship and Lemma 4, we get

$$\lim_{t\to\infty}\frac{\ddot{k}}{\dot{k}}=\lim_{t\to\infty}\left(A-\frac{\dot{c}(t)}{\dot{k}(t)}\right)=A-\lim_{t\to\infty}\frac{c(t)}{k(t)}=\lim_{t\to\infty}\frac{\dot{k}(t)}{k(t)}. \tag{11}$$

Recall also that,

$$\frac{\dot{k}(t)}{k(t)}=A-\frac{c(t)}{k(t)},$$

or, by differentiating,

$$\left(A-\frac{c(t)}{k(t)}\right)\left(\frac{\ddot{k}(t)}{\dot{k}(t)}-\frac{\dot{k}(t)}{k(t)}\right)=-\frac{c(t)}{k(t)}\left(\frac{\dot{c}(t)}{c(t)}-\frac{\dot{k}(t)}{k(t)}\right). \tag{12}$$

Since along the optimal growth path $c(t)/k(t)$ cannot converge to zero, equations (10), (11), and (12) imply

$$\lim_{t\to\infty}\frac{\dot{k}(t)}{k(t)}=\lim_{t\to\infty}\frac{\dot{c}(t)}{c(t)}=\frac{A-\theta}{\sigma}>0.$$ Q.E.D.

5. EXAMPLES

This section presents four examples/economies that illustrate the main results of the paper. Of course, the simplest balanced-growth-consistent intertemporal preference structure is one that is time-additive with constant elasticity of marginal felicity. Nevertheless, for the purpose of this paper, such a case is not interesting because intertemporal preferences play no role in explaining cross-country differences in growth rates and there are no well-defined transitional dynamics. Instead, our first two examples allow the rate of time preference to be endogenous; furthermore, they accept a balanced growth equilibrium solution and can be viewed as generalizations of the model analyzed above. Our last two examples, on the other hand, allow either for *asymptotically constant* elasticity of marginal felicity or for *asymptotically constant* discount rate and accept an *asymptotically* balanced growth equilibrium solution.

1. The first example draws on the work of Barro and Becker (1989), and Becker, Murphy, and Tamura (1990), who relate a parent's discount rate to the endogenous fertility rate. Consider the following optimization program:

$$\max\Omega=\int_0^\infty\frac{\{[c(t)]^\alpha[n(t)]^{1-\alpha}\}^{1-\sigma}}{1-\sigma}[N(t)]^{1-\varepsilon}e^{-\rho t}\,dt,$$

$$\text{subject to } \dot{k}(t)=Ak(t)-c(t)-n(t)k(t),$$

$$\dot{N}(t)=n(t)N(t),$$

$$k(0)=k_0>0,\ N(0)=N_0>0.$$

For the new variables, n is the growth rate of population, $\rho > 0$ is the constant rate-of-time preference, N is the level of population, α $(0 < \alpha < 1)$ is a parameter that captures changes in preferences towards consumption and away from children, σ is a parameter associated with the elasticity of marginal felicity,[17] and ε $(0 \leq \varepsilon \leq 1)$ is a parameter that captures preferences towards future family size and is associated with the form of the welfare function Ω; for example, if ε is equal to zero (one) then Ω becomes the Benthamite (Millian) social welfare function. Since $N(t) = N_0 \exp[-\int_0^t n(s)\,ds]$, in essence, the model allows for an endogenous subjective discount rate. More specifically, the cumulated subjective discount rate is given by $\Delta(t) = \int_0^t \{\rho - n(s)\}\,ds$. Nevertheless, as one can easily verify, the model accepts a balanced growth equilibrium path. This is because along the balanced growth path the fertility rate and hence the discount rate is constant over time. Indeed, the growth rates along the balanced growth path are[18]

$$\dot{N}(t)/N(t) = n = \frac{(\sigma-1)(1-\alpha)[\alpha(\sigma-1)A+\rho]}{\sigma[1+\alpha(\sigma-1)-\varepsilon]} > 0,$$

$$\dot{c}(t)/c(t) = \dot{k}(t)/k(t) = \frac{A-\rho}{1+\alpha(\sigma-1)} - \frac{\varepsilon(\sigma-1)(1-\alpha)[\alpha(\sigma-1)A+\rho]}{[1+\alpha(\sigma-1)]\sigma[1+\alpha(\sigma-1)-\varepsilon]}$$

$$> \frac{A-\rho}{1+\alpha(\sigma-1)} > 0.$$

These equilibrium growth rates indicate that the degree of impatience may account partially for the negative correlation between economic growth and population growth. For instance, an increase in ρ leads to a higher n but a lower rate of capital and thus lower output growth.

2. Next we provide an example where the discount rate depends on the consumption–average capital ratio, $c/\bar{k}$.[19] The representative agent seeks to maximize

$$\max \Omega = \int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\Delta(t)}\,dt,$$

$$\text{subject to } \dot{k} = Ak - c,$$

$$\dot{\Delta} = \theta(c/\bar{k}),$$

$$k(0) = k_0 > 0.$$

As the resource constraint implies, along the balanced growth path, c and k grow at the same rate and hence $\theta(c/\bar{k})$ is constant over time. Note that, although each

[17] It can be shown that $\sigma > 1$ is a necessary condition for the existence of a maximum.

[18] For further details, see Palivos and Yip (1993).

[19] This form of the discount rate conforms with the relative-income hypothesis. We thank an anonymous referee for suggesting this example.

agent takes $\bar{k}$ as given, in equilibrium $\bar{k} = k$. To find the common rate of c and k, write equations (2a) and (2b) as

$$\frac{\theta'}{\theta}\frac{1}{k}\left[\frac{c^{1-\sigma}}{\lambda(1-\sigma)} + \dot{k}\right] = \frac{c^{-\sigma}}{\lambda} - 1,$$

$$\frac{\dot{\lambda}}{\lambda} = \theta - A.$$

Combining then these two equations with the resource constraint yields[20]

$$\dot{c}(t)/c(t) = \dot{k}(t)/k(t) = \frac{A-\theta}{\sigma},$$

where $\sigma > 1$ is assumed to ensure a bounded lifetime utility. It is noteworthy that, in this example, differences in the initial consumption to capital ratio result in different growth rates, $\dot{k}(t)/k(t)$, savings ratios, $\dot{k}(t)/Ak(t)$, and wealth distributions.

3. The Stone-Geary felicity is an example of preferences with variable elasticity of marginal felicity.[21] More specifically, consider: $u[c(t)] = [c(t) - c_{\min}]^{1-\sigma}/(1-\sigma)$, $\sigma > 0$, where $c_{\min}$ is the consumption subsistence level. The elasticity of marginal felicity, is $\{1 - [c_{\min}/c(t)]\}^{-1}\sigma$, which converges, however, to σ and thus is asymptotically constant. This is the reason why an asymptotically balanced growth path exists. For example, under linear technology, it is easy to show that the growth rate of consumption is

$$\frac{\dot{c}(t)}{c(t)} = \left[1 - \frac{c_{\min}}{c(t)}\right]\frac{A-\theta}{\sigma},$$

which, is asymptotically constant, provided θ is constant over time, and converges to $(A-\theta)/\sigma$ from below. Thus, even with a linear technology, the economy displays well-defined transitional dynamics. Notice that if the economy starts close to $c_{\min}$ then it is going to experience a very long period of slow growth. This can then explain cross-country differences in growth rates without relying on differences in technology or capital immobility. In addition, differences in the desired minimum level of consumption or in initial consumption may account for different savings behavior.

4. Finally, the following example/economy allows for a variable but asymptotically constant discount rate and thus accepts an asymptotically balanced growth equilibrium. Let $u(c) = c^{1-\sigma}/(1-\sigma)$, $\theta(c) = \rho - \exp\{1/(a+c)\}$, where $\rho > 0$ and $a = 1/\log(\rho)$. Furthermore, assume that $\sigma > \max\{2, (4a^2 + 4a + 1)/4a)\}$ and

[20] Zhang (1994) utilizes a similar framework to analyze endogenous R&D decisions.
[21] This example draws on Rebelo (1992).

$\eta_k < A - [\sigma/(\sigma-1)]^{1/\sigma}(k_0)^{-1}$. One can then show that Assumptions 1–6 are satisfied.[22] Furthermore, equation (8) becomes

$$\rho - \exp\{1/(a+c)\} - A$$

$$= -\frac{\dot{c}}{c}\frac{\sigma(\sigma-1)[\rho - \exp\{1/(a+c)\}] + \dfrac{c^2\exp\{1/(a+c)\}}{(a+c)^3}\left[2 + \dfrac{1}{(a+c)}\right]}{(\sigma-1)[\rho - \exp\{1/(a+c)\}] + \dfrac{c\exp\{1/(a+c)\}}{(a+c)^2}}$$

$$+\frac{\dot{c}}{c}\frac{\dfrac{c^2\exp\{1/(a+c)\}}{(a+c)^3}\left[2 + \dfrac{1}{(a+c)}\right]\dfrac{\dot{k}/k}{A - \dot{k}/k}}{\rho - \exp\{1/(a+c)\} + \dfrac{c\exp\{1/(a+c)\}}{(a+c)^2}\dfrac{\dot{k}/k}{A - \dot{k}/k}}$$

$$-\frac{\dfrac{c\exp\{1/(a+c)\}}{(a+c)^2}\dfrac{A(\dot{k}/k)}{A - \dot{k}/k}}{\rho - \exp\{1/(a+c)\} + \dfrac{c\exp\{1/(a+c)\}}{(a+c)^2}\dfrac{\dot{k}/k}{A - \dot{k}/k}}.$$

Applying L'Hôpital's rule one can show that $\lim_{c \to 0}(\dot{c}/c) = 0$ and $\lim_{c \to \infty}(\dot{c}/c) = [A - (\rho - 1)]/\sigma$. Thus, similarly to the previous example, if the initial capital stock and hence consumption are low, there will be a very long period of slow growth. Once again, cross-country differences in growth rates can be explained without relying exclusively on differences in technology or capital immobility. Similarly to example 2, differences in the initial capital ratio will result in different growth rates as well as savings ratios and wealth distribution. In contrast to example 2, however, this model exhibits a period of transition during which there is a dynamic interaction between consumption and the degree of impatience.[23]

6. CONCLUDING REMARKS

This paper has established necessary and sufficient conditions for the existence of balanced growth and asymptotically balanced growth in an economy with variable discount rate and variable elasticity of marginal felicity, in which per capita consumption and output grow without bound. For the growth rate of the economy to be constant and hence for a balanced growth path to exist, it is necessary and sufficient

[22] Note that, under the restrictions specified above, the functional forms of $u(\cdot)$ and $\theta(\cdot)$ satisfy the sufficient conditions given in Lemma 1 in Epstein (1987).

[23] Furthermore, one can show that in a framework which combines the money-in-the-utility-function approach with variable discount rate, changes in the rate of monetary expansion will affect not only the long-run level of capital, as in Epstein and Hynes (1983), but also its growth rate.

that both the discount rate and the elasticity of marginal felicity are constant. On the other hand, for the asymptotic growth rate to be constant, these two preference parameters must be asymptotically constant. As it can be easily seen, our results hold in any perpetual growth framework, endogenous or exogenous. Nevertheless, they were obtained within an endogenous growth framework because we want to emphasize the importance of preference parameters in explaining either cross-country differences in growth rates or a single country's transition through different stages of development, as well as savings behavior and wealth distribution.

Finally, we would like to mention that, in a related study, Dolmas (1996) provides conditions that guarantee the existence of a discrete-time recursive utility model consistent with balanced growth. A crucial condition for such a balanced-growth-consistent recursive functional is the homogeneity of the aggregator function. Nevertheless, due to the fundamental differences between continuous- and discrete-time models, we are unable to contrast our necessary and sufficient conditions for the existence of a balanced growth equilibrium with the conditions found in Dolmas (1996).[24]

REFERENCES

BARRO, R. J., "Government Spending in a Simple Model of Endogenous Growth," *Journal of Political Economy* 98 (1990), S103–S125.

——— AND G. S. BECKER, "Fertility Choice in a Model of Economic Growth," *Econometrica* 57 (1989), 481–501.

BECKER, G. S. AND K. M. MURPHY, "A Theory of Rational Addiction," *Journal of Political Economy* 96 (1988), 675–700.

———, ——— AND R. TAMURA, "Human Capital, Fertility, and Economic Growth," *Journal of Political Economy* 98 (1990), S12–S37.

BECKER, R. A., "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics* 95 (1980), 375–382.

———, J. H. BOYD III AND B. Y. SUNG, "Recursive Utility and Optimal Capital Accumulation. I. Existence," *Journal of Economic Theory* 47 (1989), 76–100.

——— AND ———, "Recursive Utility and Optimal Capital Accumulation. II. Sensitivity and Duality Theory," *Economic Theory* 2 (1992), 547–563.

BOYCE, W. E. AND R. C. DIPRIMA, *Elementary Differential Equations and Boundary Value Problems*, 3rd ed. (New York: Wiley, 1977).

DEVEREUX, M. B., "Government Purchases and Real Interest Rates with Endogenous Time Preference," *Economics Letters* 35 (1991), 131–136.

DOLMAS, J., "Balanced Growth-Consistent Recursive Utility," *Journal of Economic Dynamics and Control* 20 (1996), 657–680.

DORFMAN, R., P. A. SAMUELSON AND R. M. SOLOW, *Linear Programming and Economic Analysis* (New York: McGraw-Hill, 1958).

EPSTEIN, L. G., "A Simple Dynamic General Equilibrium Model," *Journal of Economic Theory* 41 (1987), 68–95.

[24] It should be noted, however, that, within the discrete-time Uzawa–Epstein framework, the homogeneity of the aggregator function implies time-additive preferences. This follows immediately from $\Omega({}_tC) = W(c_t, \Omega({}_{t+1}C)) = u(c_t) + \Omega({}_{t+1}C)\exp\{\theta(c_t)\}$, where $W(c, \Omega)$ denotes the aggregator function.

——— AND J. A. HYNES, "The Rate of Time Preference and Dynamic Economic Analysis," *Journal of Political Economy* 91 (1983), 611–635.

GALE, D. AND W. R. SUTHERLAND, "Analysis of One Good Model of Economic Development," in G. B. Dantzig and A. F. Veinott, eds., *Mathematics of the Decision Sciences*, part 2 (Providence: American Mathematical Society, 1968), 120–136.

INTRILIGATOR, M. D., *Mathematical Optimization and Economic Theory*, (New Jersey: Prentice-Hall, 1971).

JONES, L. E. AND R. MANUELLI, "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy* 98 (1990), 1008–1038.

KALDOR, N., "Capital Accumulation and Economic Growth," in F. A. Lutz and D. C. Hague, eds., *The Theory of Capital*, (London: MacMillan, 1961), 177–222.

LUCAS, R. E. JR., "On the Mechanics of Economic Development," *Journal of Monetary Economics* 22 (1988), 3–42.

MICHEL, P., "On the Transversality Condition in Infinite Horizon Optimal Problems," *Econometrica* 50 (1982), 975–985.

NAIRAY, A., "Asymptotic Behavior and Optimal Properties of a Consumption-Investment Model with Variable Rate of Time Preference," *Journal of Economic Dynamics and Control* 7 (1984), 283–313.

OBSTFELD, M., "Aggregate Spending and the Terms of Trade: Is There a Laursen-Metzler Effect?" *Quarterly Journal of Economics* 97 (1982), 251–270.

———, "Intertemporal Dependence, Impatience, and Dynamics," *Journal of Monetary Economics* 26 (1990), 45–75.

PALIVOS, T. AND C. K. YIP, "Optimal Population Size and Endogenous Growth," *Economics Letters* 41 (1993), 107–110.

REBELO, S., "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy* 99 (1991), 500–521.

———, "Growth in Open Economies," *Carnegie-Rochester Conference Series on Public Policy* 36 (1992), 5–46.

ROMER, P. M., "Increasing Returns and Long-Run Growth," *Journal of Political Economy* 94 (1986), 1002–1037.

RYDER, H. E. JR. AND G. M. HEAL, "Optimal Growth with Intertemporally Dependent Preferences," *Review of Economic Studies* 40 (1973), 1–31.

SALA-I-MARTIN, X., "Lecture Notes on Economic Growth: Five Prototype Models of Endogenous Growth II," NBER Working paper No. 3655, 1990.

SHI, S. AND L. G. EPSTEIN, "Habits and Time Preference," *International Economic Review* 34 (1993), 61–84.

UZAWA, H., "Time Preference, the Consumption Function, and Optimal Asset Holdings," in J. N. Wolfe, ed., *Value, Capital, and Growth: Papers in Honor of Sir John Hicks* (Edinburgh: The University of Edinburgh Press, 1968), 485–504.

ZHANG, J., "An Optimal Growth Model with Endogenous R&D and Variable Discount Rate," mimeo, The University of Kansas, 1994.