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The Envelope Theorem¹

1 Introduction.

The Envelope Theorem, as presented here, is a corollary of the Karush-Kuhn-Tucker theorem (KKT) that characterizes changes in the value of the objective function in response to changes in the parameters in the problem. For example, in a standard cost minimization problem for a firm, the Envelope Theorem characterizes changes in cost in response to changes in input prices or output quantities.

There are versions of the Envelope Theorem that apply under weaker conditions and in more general settings. See, in particular, Milgrom and Segal (2002).

2 The Envelope Theorem with no binding constraints.

Consider the following parameterized version of a differentiable MAX problem with no (binding) constraints,

$$\max_{x \in \mathbb{R}^N} f(x, q)$$

where $q \in \mathbb{R}^L$ is a vector of parameters. The parametrized MIN problem is analogous. Let $\phi(q)$ give the (or at least, a) optimal x for a given q. The value function f^v is defined by

$$f^v(q) = f(\phi(q), q).$$

The following is the Envelope Theorem for this unconstrained problem. In the theorem statement and proof, $D_x f$ refers to the partial derivatives of f with respect to the x variables and $D_q f$ refers to the partial derivatives of f with respect to the q variables.

Theorem 1. Fix a parametrized differentiable MAX or MIN problem. For each $q \in \mathbb{R}^L$, let $\phi(q)$ be a solution to the MAX or MIN problem. Let $f^v : \mathbb{R}^L \to \mathbb{R}$ be defined by $f^v(q) = f(\phi(q), q)$. Fix any $q^* \in \mathbb{R}^L$ and let $x^* = \phi(q^*)$. If ϕ is differentiable at x^* , then

$$Df^{v}(q^{*}) = D_{q}f(x^{*}, q^{*}).$$

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Proof. By the Chain $Rule^2$

$$Df^{v}(q^{*}) = D_{x}f(x^{*}, q^{*})D\phi(q^{*}) + D_{q}f(x^{*}, q^{*}).$$

Since x^* is optimal when $q = q^*$, $D_x f(x^*, q^*) = 0$. The result follows.

Remark 1. Here is an alternate proof of Theorem 1 (essentially the same proof but with less reliance on intermediate results such as the fact that $D_x f(x^*, q^*) = 0$). Consider any $q^* \in \mathbb{R}^L$ and let $x^* \in \phi(q^*)$. Consider any $w \in \mathbb{R}^L$, $w \neq 0$, and any t > 0. Then, since x^* may not be optimal at $q = q^* + tw$,

$$f^{v}(q^{*} + tw) \ge f(x^{*}, q^{*} + tw).$$

Since $f^v(q^*) = f(x^*, q^*)$, it follows that,

$$f^{v}(q^{*} + tw) - f^{v}(q^{*}) \ge f(x^{*}, q^{*} + tw) - f(x^{*}, q^{*}).$$

Dividing both sides by t and taking the limit as $t \downarrow 0$ implies,

$$Df^v(q^*)w \ge D_q f(x^*, q^*)w$$

The same argument applied to -w implies $Df^{v}(q^{*})(-w) \geq D_{q}f(x^{*},q^{*})(-w)$, or,

 $Df^{v}(q^*)w \le D_q f(x^*, q^*)w,$

hence, combining,

$$Df^{v}(q^*)w = D_q f(x^*, q^*)w$$

Since this holds for any w, the result follows. \Box

Remark 2. Theorem 1 immediately generalizes to $q \in U$, where U is an open subset of \mathbb{R}^L . \Box

3 An example with no binding constraints.

Consider the cost minimization problem,

$$\begin{array}{ll} \min_{a,b\in\mathbb{R}} & a+b\\ \text{s.t.} & \sqrt{ab} \ge y\\ & a,b \ge 0 \end{array}$$

²Formally, define $r: U \to \mathbb{R}^{N+L}$ by $r(q) = (\phi(q), q)$. Then $f^{v}(q) = f(r(q))$. By the Chain Rule,

$$Df^{v}(q^{*}) = Df(x^{*}, q^{*})Dr(q^{*})$$

= $\begin{bmatrix} D_{x}f(x^{*}, q^{*}) & D_{q}f(x^{*}, q^{*}) \end{bmatrix} \begin{bmatrix} D\phi(q^{*}) \\ I \end{bmatrix}$
= $D_{x}f(x^{*}, q^{*})D\phi(q^{*}) + D_{q}f(x^{*}, q^{*}).$

as claimed. Many of the other Chain Rule applications are similar.

To interpret, a is capital, b is labor, and y is output. I assume (for simplicity) that the rental price of capital and the wage for labor both equal 1, so that total cost is a + b. If y > 0, any feasible a and b must be strictly positive, hence the non-negativity constraints do not bind at the solution. On the other hand, the production constraint $\sqrt{ab} \ge y$ binds (since cost is strictly increasing in both a and b, and since the production function is continuous), hence I can rewrite this constraint as

$$b = \frac{y^2}{a} > 0.$$

Substituting, let the (modified) objective function be

$$f(a,y) = a + \frac{y^2}{a}.$$

I can thus convert the above constrained minimization problem into an unconstrained minimization problem

$$\min_{a \in \mathbb{R}} \quad a + \frac{y^2}{a}.$$

(With the qualification that I have to check that indeed a > 0 at any proposed solution.) In terms of notation, here a is playing the role of "x" and y is playing the role of "q".

At $y = y^*$, the first order condition $D_a f(a^*, y^*) = 0$ gives,

$$1 - \frac{y^{*2}}{a^{*2}} = 0,$$

hence $y^* = y^*$ (which is strictly positive), hence $\phi(y) = a$. For this particular application, call the value function C^{ℓ} (since the value function here is what is often called *long-run cost*). Then $C^{\ell}(y) = 2y$.

Verifying the Envelope Theorem,

$$DC^{\ell}(y^{*}) = D_{y}f(a^{*}, y^{*}),$$

since

$$D_y f(a^*, y^*) = \frac{2y^*}{a^*} = 2.$$

Next, note that for a fixed (I won't try to justify here why a might be fixed), the value,

$$a + \frac{y^2}{a},$$

is the minimum cost of producing y (this is how I constructed the unconstrained problem in the first place). This can be interpreted as *short-run cost*, and accordingly I write,

$$C^s(a,y) = a + \frac{y^2}{a}.$$

Short-run and long-run cost are equal when $a = \phi(y)$: $C^{\ell}(y) = C^{s}(\phi(y), y)$. By the Envelope Theorem, then,

$$DC^{\ell}(y^*) = D_y C^s(a^*, y^*).$$

For any given value of a^* , the graph of C^s is strictly above that of C^{ℓ} (i.e., shortrun cost is strictly higher than long-run cost) except at output level y^* , since at that output level, the level of the fixed factor a is optimal. Moreover, when $y = y^*$, so that $\phi(y^*) = a^*$, the two graphs are tangent; their slopes are equal. One says that the graph of C^{ℓ} is the (lower) envelope of the graph of C^s , hence the name Envelope Theorem. Graphing C^{ℓ} and a few of the C^s will help you visualize what is going on.

All of this generalizes. If C^s is the short-run cost of producing a vector of outputs y given input prices and a fixed subvector of inputs a^* , and if a^* is optimal in the long-run when $y = y^*$, then

$$DC^{\ell}(y^*) = D_y C^s(a^*, y^*).$$

4 The Envelope Theorem with Binding Constraints.

Consider the following parameterized version of a differentiable MAX problem,

$$\max_{x} \quad f(x,q)$$

s.t. $g_{1}(x,q) \leq 0,$
 \vdots
 $g_{K}(x,q) \leq 0.$

where $q \in \mathbb{R}^{L}$ is a vector of parameters. For example, in a consumer maximization problem, the parameters are usually prices and income. The parameterized MIN problem is analogous. Again let $\phi(q)$ give the (or at least, a) optimal x for a given q and let the value function be defined by

$$f^{v}(q) = f(\phi(q), q).$$

Theorem 2 (Envelope Theorem). Fix a differentiable parameterized MAX or MIN problem. For each $q \in \mathbb{R}^L$ let $\phi(q)$ be a solution to the MAX or MIN problem. Let $f^v : \mathbb{R}^L \to \mathbb{R}$ be defined by $f^v(q) = f(\phi(q), q)$. Let J(q) be the set of indices of constraints that are binding at $\phi(q)$. Suppose that the KKT condition holds at $\phi(q)$ for every $q \in \mathbb{R}^L$.

Fix any $q^* \in \mathbb{R}^L$, let $x^* = \phi(q^*)$, let $J^* = J(q^*)$, and let $\lambda_k^* \ge 0$ be the associated KKT multipliers for $k \in J$. If ϕ is differentiable at x^* then,

$$Df^{v}(q^{*}) = D_{q}f(x^{*}, q^{*}) - \sum_{k \in J^{*}} \lambda_{k}^{*} D_{q}g_{k}(x^{*}, q^{*}).$$

Proof. By the Chain Rule (see the proof of Theorem 1),

$$Df^{v}(q^{*}) = D_{x}f(x^{*}, q^{*})D\phi(q^{*}) + D_{q}f(x^{*}, q^{*}).$$

By KKT,

$$D_x f(x^*, q^*) = \sum_{k \in J} \lambda_k^* D_x g_k(x^*, q^*).$$

Substituting,

$$Df^{v}(q^{*}) = \sum_{k \in J} \lambda_{k}^{*} D_{x} g_{k}(x^{*}, q^{*}) D\phi(q^{*}) + D_{q} f(x^{*}, q^{*}).$$

For any $k \in J^*$, let $\gamma_k(q) = g_k(\phi(q), q)$. For a MAX problem, since k is binding at q^* , $\gamma_k(q^*) = 0$ while for all other $q \in \mathbb{R}^L$, $\gamma_k(q) \leq 0$ (since $\phi(q)$ must be feasible to be a solution). Therefore, q^* maximizes γ_k on \mathbb{R}^L . (For a MIN problem, the argument is almost the same but q^* minimizes γ_k on \mathbb{R}^L .) Therefore $D\gamma_k(q^*) = 0$, hence, by the Chain Rule,

$$0 = D\gamma_k(q^*) = D_x g_k(x^*, q^*) D\phi(q^*) + D_q g_k(x^*, q^*).$$

Hence, $D_x g_k(x^*, q^*) D\phi(q^*) = -D_q g_k(x^*, q^*)$, hence, since $\lambda_k^* \ge 0$,

$$\lambda_k^* D_x g_k(x^*, q^*) D\phi(q^*) = -\lambda_k^* D_q g_k(x^*, q^*).$$

Substituting this expression for $\lambda_k^* D_x g_k(x^*, q^*) D\phi(q^*)$ into the above expression for $Df^v(q^*)$ yields the result.

Remark 3. Theorem 2 generalizes to $q \in U$, where U is an open subset of \mathbb{R}^L . \Box

5 An example with constraints.

Consider the problem

$$\max_{x} \quad \frac{1}{3}\ln(x_{1}) + \frac{2}{3}\ln(x_{2})$$

s.t.
$$p \cdot x \le m,$$
$$x \gg 0$$

where $x, p \in \mathbb{R}^{N}_{++}$ and $m \in \mathbb{R}_{++}$. Think of this as a utility maximization problem with parameters being prices $p \gg 0$ and income m > 0.

Grinding through the Kuhn-Tucker calculation, at prices p^* and m^* the solution is,

$$x^* = \begin{bmatrix} m^*/(3p_1^*) \\ (2m^*)/(3p_2^*) \end{bmatrix}.$$

Thus, the solution function ϕ is given by,

$$\phi(p,m) = \left[\begin{array}{c} m/(3p_1) \\ (2m)/(3p_2) \end{array} \right].$$

Also $J^* = \{1\}$ and

$$\lambda_1^* = \frac{1}{m^*}.$$

The value function is then determined by $f^{v}(p,m) = f(\phi(p,m))$, hence

$$f^{v}(p,m) = \frac{1}{3}\ln\left(\frac{m}{3p_{1}}\right) + \frac{2}{3}\ln\left(\frac{2m}{3p_{2}}\right)$$
$$= \ln(m) - \frac{1}{3}\ln(p_{1}) - \frac{2}{3}\ln(p_{2}) + \frac{1}{3}\ln\left(\frac{1}{3}\right) + \frac{2}{3}\ln\left(\frac{2}{3}\right).$$

Note that this makes some intuitive sense; in particular f^v is increasing in m and decreasing in p_1 and p_2 .

By direct calculation, at the point (p^*, m^*) ,

$$Df^{v}(p^{*},m^{*}) = \begin{bmatrix} \frac{\partial f^{v}}{\partial p_{1}}(p^{*},m^{*})\\ \frac{\partial f^{v}}{\partial p_{2}}(p^{*},m^{*})\\ \frac{\partial f^{v}}{\partial m}(p^{*},m^{*}) \end{bmatrix} = \begin{bmatrix} -1/(3p_{1}^{*})\\ -2/(3p_{2}^{*})\\ 1/m^{*} \end{bmatrix}$$

On the other hand, by the Envelope Theorem, since the parameters p and m don't appear in the objective function, and with the first constraint written in standard form as $g_1(x, p, m) = p \cdot x - m \leq 0$:

$$Df^{v}(p^*,m^*) = \begin{bmatrix} -\lambda_1^* x_1^* \\ -\lambda_1^* x_2^* \\ \lambda_1^* \end{bmatrix}.$$

Substitute in the value of x^* and λ^* found above and you will confirm that these two expressions for $Df^v(p^*, m^*)$ are indeed equal.

6 The constraint term and the interpretation of the λ_k .

Consider a MAX problem and focus on K parameters with the following special form. Parameter q_k affects only constraint k and it does it additively:

$$g_k(x) - q_k \le 0.$$

Increasing q_k weakens the constraint. Then, from the Envelope theorem, at $q_k^* = 0$,

$$D_{q_k}f^v(0) = -\lambda_k^*(-1) = \lambda_k^*$$

for any $k \in J^*$. That is, λ_k^* is the marginal value of relaxing binding constraint k. The argument for MIN problems is almost identical.

For $k \notin J$, set $\lambda_k^* = 0$. For some versions of KKT, this is actually a requirement of KKT. For KKT as I have stated it, it is more in the nature of a convention. In any event, the interpretation is that if constraint k is not binding then the marginal value of relaxing constraint k is zero.

References

Milgrom, Paul and Ilya Segal, "Envelope Theorems for Arbitrary Choice Sets," Econometrica, 2002, 70 (2), 583–601.