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# The Revenue Equivalence Theorem for Basic Auctions[1]

## 1 Introduction.

These notes concern auctions in which a single unit is for sale and in which bidders have valuations that are private (no bidder's own valuation could be affected by the information held by other bidders) and independent (no bidder can use their own valuation to gain insight into the valuations, and hence bidding behavior, of the other bidders).

For this setting, the Revenue Equivalence Theorem gives conditions under which some very different auctions generate the same expected revenue. I discuss whether revenue equivalence holds in more general settings very briefly in Section 6. A special case of revenue equivalence was first stated and proved in Vickrey (1961) and Vickrey (1962). General revenue equivalence results appeared in Myerson (1981) and Riley and Samuelson (1981).

The argument underlying revenue equivalence plays a role in some other important results in game theory, including the Myerson-Satterthwaite theorem, Myerson and Satterthwaite (1983), which states that, in a broad class of settings, inefficiency in bargaining is an unavoidable consequence of asymmetric information. For textbook treatments of revenue equivalence, both in auctions and other settings, see Krishna (2009), Milgrom (2004), and Borgers, Krahmer and Strausz (2015).

## 2 Examples.

A seller owns one unit of an indivisible object. $N \geq 2$ other people, the *bidders*, are interested in acquiring the object. Bidder $i$ values the object at $v_i$, which I also refer to as the bidder's *value* or *type* (I use these terms interchangeably). The $v_i$ are distributed independently and uniformly on $[0, 1]$. Bidder $i$ knows $v_i$ but not the values of the other bidders. The distribution of the $v_i$ is common knowledge.

Consider the following four auctions. In all four auctions, each bidder $i$ submits a bid $b_i \geq 0$. And in all four auctions, the high bidder wins. If there is more than one highest bid then a referee chooses a winner at random from among the highest bidders with equal probability. (Ties are, in fact, a zero probability event in equilibrium in all four examples.) Where the auctions differ is in the payment rules.

1. *First Price Sealed Bid Auction (1P)*. The winning bidder pays her bid, for a net utility of
$$v_i - b_i.$$
The other bidders pay nothing, for a net utility of 0.

   One can compute that the unique Nash equilibrium of 1P is for each bidder $i$ to bid
$$b_i = v_i - \frac{v_i}{N}.$$
Thus, the bidder bids less than the true value ("shaves" her bid), but her bid is close to the true value if $N$ is large.

2. *Second Price Sealed Bid Auction (2P)*. The winner pays the *second* highest bid (which equals the highest bid if there are two or more bids tied for highest bid; but this is a zero probability event in equilibrium). The other bidders pay nothing, for a net utility of 0.

   2P has a Nash equilibrium in which each bidder bids her true value.
$$b_i = v_i.$$
One can check that this bid is actually weakly dominant: a deviation from this bid never helps the bidder and sometimes hurts him.[2]

3. *All Pay Auction.* All bidders pay their bids.

   There is an equilibrium in which the bid is,
$$b_i = (v_i)^N - \frac{(v_i)^N}{N}.$$
For $v_i \in (0, 1)$, this is strictly smaller than the equilibrium bid in 1P, and the difference between the two is larger the larger is $N$. Indeed, even for $v_i$ very close to one, a bidder in the All Pay auction will bid close to 0 if $N$ is large.

4. *Third Price Sealed Bid Auction.* Assume $N \geq 3$. The winner pays the *third* highest bid (which equals the highest bid if three or more bids tie for the highest bid; but this a zero probability event in equilibrium). The other bidders pay nothing, for a net utility of 0.

   There is an equilibrium in which the bid is,
$$b_i = v_i + \frac{v_i}{N}.$$
This bid is *larger* than the bidder's value, although it is close to the bidder's value if $N$ is large.

[2] There are other Nash equilibria. For example, it is an equilibrium for bidder 1 to bid 1 and for all other bidders to bid 0. But I will focus on the truthful equilibrium.

Now for the punchline: in the given Nash equilibria of all four auctions, expected revenue to the seller is the same, namely,

$$\frac{N-1}{N+1}.$$

The claim is *not* that the auctions generate the same revenue for every realization of values. For example, if $N = 2$ and the values are $v_1 = 3/4$ and $v_2 = 1/2$ then the 1P auction generates revenue of $(3/4)/2 = 3/8$ while the 2P auction generates revenue of $1/2$: the 2P auction generates higher revenue. But if $v_1 = 3/4$ and $v_2 = 1/4$ then the 1P auction still generates revenue of $3/8$ while the 2P the auction generates revenue of $1/4$: the 1P auction generates higher revenue. Revenue equivalence says that these differences wash out in expectation.

# 3 Auctions.

## 3.1 The information structure.

A seller owns one unit of an indivisible object. $N$ other people, the *bidders*, are interested in acquiring the object. Bidder $i$ values the object at $v_i \geq 0$, called the bidder's *value* or *type* (I use these terms interchangeably).

Let $F$ be the joint distribution over *type profiles* $v = (v_1, \ldots, v_N)$. Let $\text{supp}(F)$ denote the support of $F$: $\text{supp}(F)$ is the smallest closed set that gets probability 1. For each $i$, let $V_i$ be the set of $v_i$ that appear in $\text{supp}(F)$:

$$V_i = \{v_i \geq 0 : \exists\, v_{-i} \text{ s.t. } (v_i, v_{-i}) \in \text{supp}(F)\},$$

where $v_{-i}$ is a type profile for bidders other than $i$ and $(v_i, v_{-i})$ is the type profile determined by $v_i$ and $v_{-i}$. Since $v_i$ has a lower bound (namely 0) and $V_i$ is closed, $V_i$ has a smallest element, $\underline{v}_i$. I assume that $V_i$ is also bounded above and hence has a largest element, $\overline{v}_i$. Thus $V_i \subseteq [\underline{v}_i, \overline{v}_i]$. The Revenue Equivalence Theorem assumes that, in fact, $V_i = [\underline{v}_i, \overline{v}_i]$. The analysis extends easily to cases in which $V_i$ is not bounded above, provided the required expectations are finite.

If $F$ is independent, then $\text{supp}(F)$ can be written as the product $\text{supp}(F) = V_1 \times \cdots \times V_N$. The Revenue Equivalence Theorem assumes independence but some of the definitions below do not. Section 6.2 shows by example that Revenue Equivalence can fail if independence is dropped.

*Remark* 1. Statements of the Revenue Equivalence Theorem often require that the marginals have densities: no $v_i$ has positive probability. The density assumption is not, in fact, needed for the theorem itself. The density assumption does, however, sometimes plays a role in applications. □

Bidder $i$ knows $v_i$ but not the values of the other bidders. I assume, however, that $F$ is common knowledge (it is part of the structure of the game).

Henceforth, hold this basic information structure fixed.

### 3.2 An auction game form.

Now consider a specific auction. In this auction, bidder $i$ can take an action $a_i \in A_i$. The set $A_i$ is non-empty but otherwise unrestricted. Let $A = \prod A_i$; an element of $a \in A$, $a = (a_1, \ldots, a_N)$, is an *action profile* .

In some auctions, $a_i$ could be just a number, the bid, as was the case in the auctions described above. But in other auctions, $a_i$ could be complicated. The auction could, for example, be an English (open-outcry ascending price) auction, commonly used for art: bids are observable and, at each instant, any bidder can enter a new bid, provided the new bid is higher than the highest previous bid. $a_i$ is then the name of the function that specifies what bidder $i$ does as a function of time and the history of prior bids.

Since a bidder observes her own value but not those of the other bidders, a bidder can condition her action on her own value, but she cannot condition on the values of the other bidders. A pure strategy for bidder $i$ in the auction is therefore a function,

$$s_i : V_i \to A_i.$$

I denote by $s$ the *strategy profile* $(s_1, \ldots, s_N)$. Abusing notation somewhat, let $s(v) = (s_1(v_1), \ldots, s_N(v_N))$.

### 3.3 Utility.

I assume that bidders are risk neutral. I discuss risk aversion briefly in Section 6.4.

Action profile $a$ determines outcomes via functions $\pi_i : A \to [0, 1]$ and $\tau_i : A \to \mathbb{R}$.

$\pi_i(a)$ is the probability that bidder $i$ gets the object. For any $a \in A$, I require that

$$\sum_i \pi_i(a) \leq 1.$$

I allow $\sum_i \pi_i(a) < 1$: the seller could retain the object.

$\tau_i(a)$ is the payment made by the bidder. I do not assume that $\tau_i(a)$ is positive, although this will be the case in typical auctions.

Finally, utilities depend on types and on outcomes.[3] In particular, if bidder $i$ has value $v_i$ and the bid profile is $a$ then bidder $i$ wins with probability $q_i = \pi_i(a)$ and pays $t_i = \tau_i(a)$; bidder $i$'s expected utility in this case is,

$$u_i(v_i, q_i, t_i) = q_i v_i - t_i.$$

This specification of $\tau_i$ appears to rule out auctions in which bidder $i$ does not pay if she does not win. Such auctions have not, in fact, been ruled out. One merely has to interpret the payment as an *expected* payment. Fix $a$ and let $q_i = \pi_i(a)$. Suppose

[3]In game theory, it is standard to refer to utility as the payoff. I use the term "utility", rather than "payoff", to avoid confusion with the similar-sounding "payment".

that bidder $i$ makes the payment $\hat{t}_i$ if she gets the object and $\tilde{t}_i$ (which could be 0) if she does not. In expectation, therefore, she pays $t_i = q_i\hat{t}_i + (1 - q_1)\tilde{t}_i$. This notational trick works if, as I assume, preferences are risk neutral.

Note also that there *are* auctions where the bidder pays even if the bidder does not win. This was true for the All Pay auction discussed in Section 2, and it is true for raffles, discussed in the next remark.

*Remark* 2. As an example of an "auction" in which $q_i$ is typically strictly between 0 and 1, consider a *raffle*. The action $a_i \in \mathbb{R}_+$ represents the purchase of $a_i$ raffle tickets. Assuming at least one person buys at least one raffle ticket,

$$q_i = \frac{a_i}{\sum_{j=1}^{N} a_j},$$

and

$$t_i = a_i.$$

□

### 3.4 Nash equilibrium.

In the auction game, nature moves first and selects a value profile $v$ according to the joint distribution $F$. In this game, a strategy profile $s$ is a (pure strategy) *Nash equilibrium (NE)* iff, for each $i$ and each $v_i \in V_i$, $s_i(v_i)$ maximizes bidder $i$'s expected utility given the strategies of the other bidders. The expectation is with respect to the other bidders' values, $v_{-i}$, which bidder $i$ does not know. It is customary to call a NE in this sort of information setting a *Bayesian Nash Equilibrium.*[4]

## 4 Revenue Equivalence.

Fix a strategy profile $s$ (not necessarily an equilibrium profile). Define $Q_i : V_i \to [0, 1]$ by,

$$Q_i(v_i) = \mathbb{E}_{v_{-i}}[\pi_i(s(v_i, v_{-i}))|v_i],$$

where $v_{-i}$ is the vector of values of the other bidders and $(v_i, v_{-i})$ is the vector of all the values, including $v_i$. $Q_i(v_i)$ is the probability that $i$ gets the good, given the strategy profile $s$ and given $i$'s own type $v_i$. Similarly, define $T_i : V_i \to \mathbb{R}$ by,

$$T_i(v_i) = \mathbb{E}_{v_{-i}}[\tau_i(s(v_i, v_{-i}))|v_i].$$

[4]Strictly speaking, the equilibrium concept here is somewhat stronger than standard Nash equilibrium. Under standard Nash equilibrium, $s_i$ is required to be optimal only *ex ante*: $s_i$ maximizes the bidder $i$'s expected utility given the strategies of the other bidders, where the expectation is over $v$. This is equivalent to requiring that $s_i(v_i)$ maximize $i$'s expected utility for a subset of $v_i$ of probability one, where the expectation is over $v_{-i}$. If, for example, the marginal distribution over $V_i$ has a density, so that each individual value of $v_i$ has probability zero, then $s_i$ could satisfy the standard NE optimization condition even if $s_i(v_i)$ is not optimal for some $v_i$. I will, implicitly, rule this sort of thing out.

$T_i(v_i)$ is bidder $i$'s expected payment, given the strategy profile and given $i$'s own type. Thus, if the strategy profile is $s$, then bidder $i$, with value $v_i$, has an expected utility of,

$$U_i(v_i) = Q_i(v_i)v_i - T_i(v_i).$$

In words, a bidder's type influences her expected utility through three channels. There is a direct affect: $v_i$ is the value of the object if bidder $i$ wins. There are also two indirect effects. First, $v_i$ determines $i$'s action $a_i$ via the strategy $s_i$. Second, the expectations used for $Q_i$ and $T_i$ are conditional on $v_i$. Bidder $i$, if she has a high $v_i$, could infer that the values of the other bidders are likely to be high as well, and this inference affects her posterior over what actions the other bidders will take. However, the Revenue Equivalence Theorem assumes this second indirect effect away, by assuming that $F$, the joint distribution over bidder values, is independent. Section 6.2 discusses the independence assumption further.

The key fact underlying Revenue Equivalence is the following, which I prove in Section 5.

**Theorem 1** (The Integral Condition)**.** *Suppose that $F$ is independent and that, for each $i$, $V_i$ is an interval, $V_i = [\underline{v}_i, \overline{v}_i]$. Consider any strategy profile $s$. If $s_i$ is optimal for bidder $i$, then, for any $v_i \in V_i$,*

$$U_i(v_i) = U_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} Q_i(x)\, dx. \tag{1}$$

Equation (1) is often called the *Integral Condition.* Since $T_i(v_i) = Q_i(v_i)v_i - U_i(v_i)$, the integral condition implies that,

$$T_i(v_i) = Q_i(v_i)v_i - U_i(\underline{v}_i) - \int_{\underline{v}_i}^{v_i} Q_i(x)\, dx. \tag{2}$$

This says that bidder $i$'s expected payment when her value is $v_i$ depends only on $U_i(\underline{v}_i)$ and $Q_i$. Since $T_i(v_i)$ is bidder $i$'s expected payment, it is the seller's expected revenue. Therefore, in a NE, since every bidder is optimizing, the seller's overall expected revenue is,

$$\mathbb{E}_v\left[T_1(v_1) + \cdots + T_N(v_N)\right]. \tag{3}$$

**Theorem 2** (The Revenue Equivalence Theorem)**.** *Suppose that $F$ is independent and that, for each $i$, $V_i$ is an interval, $V_i = [\underline{v}_i, \overline{v}_i]$. Consider any two auctions and any two NE of those auctions. If $Q_i$ and $U_i(\underline{v}_i)$ are the same for each $i$ in both NE, then the expected revenue to the seller is the same in both NE.*

**Proof.** Almost immediate from Equation 3 since, in NE, all bidders are optimizing, and since, by Equation 2, expected revenue from an optimizing bidder $i$ depends only on $Q_i$ and $U_i(\underline{v}_i)$. ■

In all four auction examples in Section 2, $\underline{v}_i = 0$ and $U_i(0) = 0$: if you have the lowest possible value, you expect to get nothing in the auction. And in all four auction examples, the bidder with highest $v_i$ wins, hence, for every $i$, $Q_i$ is the same.[5] Since $U_i(\underline{v}_i)$ and $Q_i$ are the same in all four auctions, the Revenue Equivalence theorem implies that the expected revenue must be the same, as was indeed the case.

## 5 Proof of Theorem 1, the Integral Condition.

Define $\tilde{Q}_i : V_i \times V_i \to [0, 1]$ by,

$$\tilde{Q}_i(v_i, v_i^*) = \mathbb{E}_{v_{-i}}[\pi_i(s(v_i, v_{-i}))|v_i^*].$$

$\tilde{Q}_i(v_i, v_i^*)$ is the probability that bidder $i$ wins if her true value is $v_i^*$ but she acts as if her type were $v_i$ and takes action $a_i = s_i(v_i)$. Define $\tilde{T}_i$ analogously.

The assumption that $F$ is independent implies that,

$$\tilde{Q}_i(v_i, v_i^*) = Q_i(v_i),$$

and,

$$\tilde{T}_i(v_i, v_i^*) = T_i(v_i).$$

Therefore, given independence, bidder $i$'s expected utility when she is of type $v_i^*$ but plays as if she were of type $v_i$ is,

$$\tilde{U}_i(v_i, v_i^*) = Q_i(v_i)v_i^* - T_i(v_i).$$

Note that $U_i(v_i^*) = \tilde{U}(v_i^*, v_i^*)$. In Section 6.2, I discuss what happens when independence is violated.

If $s_i$ is optimal, then, for any $v_i$ and $v_i^*$, it must be that

$$U_i(v_i^*) = \tilde{U}(v_i^*, v_i^*) \geq \tilde{U}(v_i, v_i^*). \tag{4}$$

Conversely,

$$U_i(v_i) = \tilde{U}(v_i, v_i) \geq \tilde{U}(v_i^*, v_i). \tag{5}$$

These inequalities are called *incentive compatibility (IC)* conditions. The IC inequalities imply,

$$\begin{aligned} U_i(v_i^*) - U_i(v_i) &\geq Q_i(v_i)v_i^* - T_i(v_i) - U_i(v_i) \\ &= Q_i(v_i)v_i^* - T_i(v_i) - [Q_i(v_i)v_i - T_i(v_i)] \\ &= Q_i(v_i)(v_i^* - v_i). \end{aligned}$$

[5]Explicitly, since the marginal distributions have densities, the probability that two or more $v_i$ are tied for being highest is zero. Therefore, $Q_i(v_i)$ equals the probability that $v_i$ is strictly higher than any $v_j$, $j \neq i$.

Similarly,

$$\begin{aligned}U_i(v_i^*) - U_i(v_i) &\leq U_i(v_i^*) - [Q_i(v_i^*)v_i - T_i(v_i^*)] \\ &= Q_i(v_i^*)v_i^* - T_i(v_i^*) - [Q_i(v_i^*)v_i - T_i(v_i^*)] \\ &= Q_i(v_i^*)(v_i^* - v_i).\end{aligned}$$

Combining,

$$Q_i(v_i)(v_i^* - v_i) \leq U_i(v_i^*) - U_i(v_i) \leq Q_i(v_i^*)(v_i^* - v_i). \tag{6}$$

If $v_i^* > v_i$, this implies $Q_i$ is weakly increasing in $v_i$: as $v_i$ increases, it is increasingly likely that bidder $i$ will get the object.

Since the domain of $Q_i$, namely $V_i$, is an interval and since $Q_i$ is weakly increasing, $Q_i$ is (Riemann) integrable (Rudin (1976), Theorem 6.9). Inequality (6) implies that the integral satisfies,

$$U_i(v_i) - U_i(\underline{v}_i) = \int_{\underline{v}_i}^{v_i} Q_i(x)\, dx, \tag{7}$$

where $\underline{v}_i$ is the minimum value of $v_i$.[6]

Rearranging Equation (7) gives the Integral Condition, Equation (1), which proves Theorem 1. ■

*Remark* 3. Assume for the moment that $Q_i$ and $T_i$ are differentiable. As in the proof of Theorem 1, let $\tilde{U}_i(v_i, v_i^*)$ be the expected utility to a bidder with value $v_i^*$ when she plays action $a_i = s_i(v_i)$. Thus, again as in the proof of Theorem 1, assuming independence,

$$\tilde{U}_i(v_i, v_i^*) = Q_i(v_i)v_i^* - T_i(v_i).$$

Consider the problem of maximizing $\tilde{U}_i(v_i, v_i^*)$ with respect to $v_i$ (the type that bidder $i$ is pretending to be), holding the true type, $v_i^*$, fixed. If $s_i$ is indeed optimal, then the solution is to set $v_i = v_i^*$ (so that bidder $i$ plays action $a_{*i} = s_i(v_i^*)$). $U_i$ is then the value function for this maximization problem. By the Envelope Theorem, $DU_i(v_i^*) = D_2\tilde{U}_i(v_i^*, v_i^*)$ (where $D_2$ is the derivative with respect to the second argument), hence

$$DU_i(v_i^*) = Q_i(v_i^*). \tag{8}$$

[6] Explicitly, assume that $v_i > \underline{v}_i$ (if not, then the argument becomes trivial). Recall that a function is integrable iff its upper and lower integrals are equal (and finite). Consider any finite partition of $[\underline{v}_i, v_i]$ determined by $\underline{v}_1 = x_1 < \cdots < x_K = v_i$. Consider any $x_k < x_{k+1}$. Since $Q_i$ is weakly increasing, the maximum value of $Q_i$ on this interval is $Q_i(x_{k+1})$, so that the associated rectangle has area $Q_i(x_{k+1})(x_{k+1} - x_k)$. By (6), this area is bounded below by $U_i(x_{k+1}) - U_i(x_k)$. Summing over all partition intervals, all but two of the $U_i(x_k)$ cancel, yielding a total lower bound of $U_i(v_i) - U_i(\underline{v}_i)$. I get this same lower bound for every partition. Hence the upper integral is bounded below by $U_i(v_i) - U_i(\underline{v}_i)$. By an analogous argument, the lower integral is bounded above by $U_i(v_i) - U_i(\underline{v}_i)$. Since $Q_i$ is integrable, this yields Equation (7).

Integrating Equation (8) this gives Equation (7).

In terms of the three channels by which $v_i$ can affect $U_i$ (see Section 4), the $Q_i(v_i^*)$ in Equation (8) is from the direct channel. The assumption that the joint distribution is independent has killed off the second indirect channel (conditional expectation). Section 6.2 discusses what happens to Equation (8) if independence is relaxed. Finally, the Envelope Theorem says that, because of optimization, the first indirect channel (choice of action as a function of type), has no first-order affect on utility.

One cannot, in fact, simply assume that $Q_i$ and $T_i$ are differentiable, since they depend on the strategies, which are not required to be differentiable. Milgrom and Segal (2002) shows that, despite this issue, a generalized envelope theorem argument still applies. $\square$

*Remark* 4. Inequalities (4) and (5) are often called *incentive compatibility* conditions. $\square$

# 6 Qualifications and Extensions.

## 6.1 If the $V_i$ are not intervals.

Revenue equivalence can fail if the $V_i$ are not intervals. As an example, suppose that $N = 2$ and that the $v_i$ are equally likely to be 0 or 1, independently. Hence, $V_1 = V_2 = \{0, 1\}$ (not $[0, 1]$).

In this environment, the 2P auction still has an equilibrium in which bidders bid their true values; in that equilibrium, the object goes to the bidder with highest value, and the expected utility of a bidder of lowest value is zero. Revenue in this auction is zero except when both bidders have values $v_i = 1$, in which case revenue is 1. Since the probability that both bidders have value 1 is 1/4, expected revenue is 1/4.

But the seller can do better with the following "auction", which I call PP (posted price). The object is given to a bidder who says "buy". If no bidder says "buy", the seller keeps the object. (If both bidders say "buy", each bidder is equally likely to get the object.) If there is a winner, the winner pays 2/3. In this auction, the (unique) equilibrium has each bidder say "buy" iff her value is 1.

In both equilibria of both auctions, $u_i(\underline{v}_i) = u_i(0) = 0$. And both have the same $Q_i$ (for $i = 1$: $Q_1(0) = 0$; $Q_1(1) = 3/4$, which is the probability that $v_2 = 0$ plus one half times the probability that $v_2 = 1$). Therefore, if Revenue Equivalence held, expected revenue in both would be 1/4. But in the PP auction, expected revenue is 1/2.

A partial intuition for what is going on here is that any equilibrium of any auction must satisfy the incentive compatibility (IC) conditions, Inequality (4) and Inequality (5). Very loosely, the more IC conditions there are, the more restricted is equilibrium behavior. If the $V_i$ are intervals, then bidder $i$ has, for each realized

$v_i$, a continuum of IC conditions. In contrast, if $V_i = \{0, 1\}$, then bidder $i$ has, for each realized $v_i$, only a single IC condition (e.g., it cannot be profitable for a $v_i = 1$ bidder to behave as if $v_i = 0$, or vice versa).

In terms of the formal argument, the Integral Condition, Equation (1), assumes that the relevant functions are all defined on an interval. And the Envelope Theorem condition (Equation (8) in Remark 3 in Section 5) assumes that the relevant expectations are defined for types arbitrarily near the true type.

### 6.2 If $F$, the joint type distribution, is not independent.

Revenue Equivalence can fail if the joint type distribution is not independent. As an example, suppose that $N = 2$, that $v_1$ is distributed uniformly on $[0, 1]$, and that $v_2 = 1 - v_1$. Here, $V_1 = V_2 = [0, 1]$.

The 2P auction still has an equilibrium in which bidders bid their true value. The expected revenue from this equilibrium is the expected value of the low value bidder, namely 1/4.

But the seller can do better with the following auction. Each bidder $i$ bids a number $b_i \in [0, 1]$. If the numbers satisfy $b_2 = 1 - b_1$, then the high bidder gets the object and pays 1/2. Otherwise, the seller keeps the object and no money changes hands. This auction also has an equilibrium in which bidders bid their true values.

In both equilibria of both auctions, $u_i(\underline{v}_i) = u_i(0) = 0$. And both have the same $Q_i$ ($Q_i(v_i) = 0$ if $v_i < 0$; $Q_i(v_i) = 1$ if $v_i > 0$; $Q_i(1/2) = 1/2$, but this last case has zero probability of occurring). Therefore, if Revenue Equivalence held, expected revenue in both would be 1/4. But in the new auction, which exploits the negative correlation in values, the expected revenue is 1/2.

In terms of the formal argument, if $F$ is not independent then, in the notation used in the proof of Theorem 1 in Section 5, it is no longer necessarily true that $\tilde{Q}(v_i, v_i^*) = Q_i(v_i)$ and $\tilde{T}(v_i, v_i^*) = T_i(v_i)$. This vitiates the subsequent analysis because the expected payment terms may not drop out. In terms of the Envelope Theorem approach (Remark 3 in Section 5), Equation (8) becomes,

$$DU_i(v_i^*) = \tilde{Q}_i(v_i^*, v_i^*) + D_2\tilde{Q}_i(v_i^*, v_i^*) - D_2\tilde{T}_i(v_i^*, v_i^*).$$

Note that there is now a payment term on the right-hand side.

### 6.3 Asymmetric type distributions.

If the $v_i$ are drawn from different distributions then the equilibria of a 1P auction may no longer have the property that the highest valuation bidder wins the object. As a consequence, Revenue Equivalence between the 1P and 2P auctions can break down. The ranking could go either way: the 1P auction could generate more expected revenue than the 2P auction or vice versa, depending on the distribution of the $v_i$.

### 6.4 Risk aversion.

If bidders are risk averse, meaning that utility is a concave function of $v_i$ less the bidder's payment, then Revenue Equivalence can fail. Consider, in particular, the symmetric setting of Section 2. If bidders are risk averse, then the 1P auction generates higher expected revenue than the 2P auction, even though both auctions still assign the object to the high valuation bidder. The intuition is that it remains an equilibrium in the 2P auction for bidders to bid their type, but risk aversion causes bidders in a 1P auction to shave their bids by less than under risk neutrality, and under risk neutrality, the expected revenues of the two auctions were equal. (If we apply a concave function only to $v_i$, then Revenue Equivalence continues to hold.)

### 6.5 Interdependent values.

Suppose that values are interdependent in the sense that, rather than each bidder knowing $v_i$, each bidder receives a signal about $v_i$, and $v_i$ could also depend on the signals received by the other bidders. Two special cases are the following.

- *Private values.* Bidder $i$'s value is her signal; her value does not depend directly on the signals of the other bidders. Bidder $i$ cares about the signals of the other bidders to the degree that these signals affect the behavior of the other bidders, but that is a separate issue. Private values is the assumption in the Revenue Equivalence Theorem as stated.

- *Common values.* All bidders value the object the same, but they have different signals as to that value.

The classic citation for auctions with interdependent values is Milgrom and Weber (1982).

If the value to bidder $i$ is additively separable in the signals, and if the joint distribution over signals is independent, then Revenue Equivalence continues to hold. More generally, however, Revenue Equivalence may fail. In particular, in many settings, the 2P auction can generate higher expected revenue than the 1P auction for the following reason. In an auction with interdependent values, winning can be bad news in the sense that the winner infers that the other bidders must have received less promising signals of the object's true value. This is called the *winner's curse*. In a 1P auction, anticipation of the winner's curse gives each bidder an additional reason to shave her bid, lowering overall expected revenue. In contrast, the winner's curse is less of an issue in the 2P auction precisely because bidders do not pay their own bids.

### 6.6 Generalization.

Finally, see Milgrom and Segal (2002) for a discussion of how Revenue Equivalence extends to more general environments (for example, environments with multiple goods).

# 7 An Application: Computing the Equilibria of Simple Auctions.

Consider, as an Section 2, a first price (1P) auction in which the bidder values are all distributed uniformly on $[0, 1]$. Conjecture that there is a symmetric equilibrium in which bids are strictly increasing in bidder value. Then $Q_i(v_i)$, the probability of winning when of type $v_i$, is the probability that $v_i$ is the highest value

$$Q_i(v_i) = [v_i]^{N-1}.$$

In a 1P auction, the winning bidder pays her own bid, $b_i$, so the expected payment by a bidder of type $v_i$ is $b_i Q_i(v_i)$, hence

$$T_i(v_i) = b_i [v_i]^{N-1}.$$

On the other hand, we know from Equation 2 that (assuming $U_i(\underline{v}_i) = 0$, as will in fact be the case),

$$\begin{aligned} T_i(v_i) &= Q_i(v_i)v_i - U_i(\underline{v}_i) - \int_0^{v_i} Q_i(x)\, dx \\ &= [v_i]^{N-1}v_i - \int_0^{v_i} x^{N-1}\, dx \\ &= [v_i]^N - \frac{[v_i]^N}{N}. \end{aligned}$$

Combining,

$$b_i = v_i - \frac{v_i}{N},$$

as stated in Section 2. (If $v_i = 0$ then it is weakly dominant to bid $b_i = 0$.) Note that the bids are indeed symmetric and strictly increasing in bidder value, as conjectured.

On the other hand, for the all pay (AP) auction, the bidder always pays $b_i$. Conjecture again that there is a symmetric equilibrium in which bids are strictly increasing in bidder value. Then we immediately get

$$b_i = T_i(v_i) = [v_i]^N - \frac{[v_i]^N}{N},$$

as stated in Section 2. Again, note that the bids are indeed symmetric and strictly increasing in bidder value, as conjectured.

Revenue Equivalence also provides a relatively easy way to compute the expected revenue of any of the four auctions in Section 2: the expected revenue must be the same as the expected revenue from the 2P auction, and that is simply the expected value of the second highest type.

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