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$\mathbb{R}^N$ Completeness and Compactness[1]

1 Completeness in $\mathbb{R}$.

As a preliminary step, I first record the following compactness-like theorem about $\mathbb{R}$.

Theorem 1 (Bolzano-Weierstrass in $\mathbb{R}$)**.** *If a sequence in $\mathbb{R}$ is bounded then it has a convergent subsequence.*

Proof. Let (x_t) be a sequence in $\mathbb{R}$. Let E be the range of (x_t). If E is finite then some value is repeated infinitely often; take the subsequence corresponding to that value. This subsequence converges, trivially.

Otherwise, suppose that E is infinite. Since (x_t) is bounded, there are $a_0 < b_0$ such that $E \subseteq [a_0, b_0]$. Let m_0 be the midpoint of $[a_0, b_0]$: $m_0 = (b_0 + a_0)/2$. Since E is infinite, either $E \cap [a_0, m_0]$ is infinite or $E \cap [m_0, b_0]$ is infinite, or both. If the former, take $a_1 = a_0$ and $b_1 = m_0$. Otherwise, take $a_1 = m_0$, $b_1 = b_0$. Either way, $b_1 > a_1$.

And so on. At stage t, $b_t > a_t$ and $E \cap [a_t, b_t]$ is infinite. Let m_t be the midpoint of $[a_t, b_t]$: $m_t = (b_t + a_t)/2$. Since $E \cap [a_t, b_t]$ is infinite, either $E \cap [a_t, m_t]$ is infinite or $E \cap [m_t, b_t]$ is infinite, or both. If the former, take $a_{t+1} = a_t$ and $b_{t+1} = m_t$. Otherwise, take $a_{t+1} = m_t$, $b_{t+1} = b_t$. Either way, $b_{t+1} > a_{t+1}$.

We thus get sequences (a_t) and (b_t) with $a_0 \leq a_1 \leq \cdots \leq b_1 \leq b_0$; in particular, $a_s < b_t$ for any s and any t. Let A be the range of (a_t) and let B be the range of (b_t). By the Least Upper Bound (LUB) property, there is an $a^* = \sup A$ and a $b^* = \inf B$. Moreover, $a^* \leq b^*$.[2]

Since $a_t \leq a^* \leq b^* \leq b_t$ for every t, it follows that $b^* - a^* \leq b_t - a_t = (b_{t-1} - a_{t-1})/2 = \cdots = (b_0 - a_0)/2^t$ for all t, hence $a^* = b^*$. Call this common value x^*. Note that $x^* \in [a_t, b_t]$ for all t.

Finally, construct a subsequence (x_{t_k}) such that, for each k, $x_{t_k} \in [a_k, b_k]$. This is possible since, for each k, $E \cap [a_k, b_k]$ is infinite, hence one can always find an index t_k such that $t_k > t_{k-1}$ and $x_{t_k} \in [a_k, b_k]$. Then $x_{t_k} \to x^*$ since, for any k, both x_{t_k} and x^* lie in $[a_k, b_k]$ and hence $|x_{t_k} - x^*| \leq b_k - a_k = (b_0 - a_0)/2^k$. ■

[2] Explicitly, for every s, t, $a_s < b_t$, hence any a_s is a lower bound of B. Since b^* is the *largest* lower bound of B, $a_s \leq b^*$ for every s, hence b^* is an upper bound of A. Since a^* is the *smallest* upper bound of A, $a^* \leq b^*$.

It is worth emphasizing that Theorem 1 relies heavily on the Least Upper Bound (LUB) property of $\mathbb{R}$. The theorem is false, in particular, if instead of $\mathbb{R}$ we are working with $\mathbb{Q}$ (as usual, just consider any sequence (x_t) of rational numbers that converges, in $\mathbb{R}$, to an irrational number x^*; then (x_t) is Cauchy, since it is convergent in $\mathbb{R}$, but since x^* is not in $\mathbb{Q}$, no subsequence converges if the space is restricted to $\mathbb{Q}$).

Theorem 2. $\mathbb{R}$ *is complete.*

Proof. Let (x_t) be a Cauchy sequence in $\mathbb{R}$. Then (x_t), being Cauchy, is bounded. By Theorem 1, (x_t) then has a convergent subsequence. Finally, any Cauchy sequence with a convergent subsequence is convergent, by a theorem in the notes on Metric Spaces. ■

It also follows immediately from Theorem 1 that any set in $\mathbb{R}$ that is closed as well as bounded is sequentially compact; this is the Heine-Borel Theorem in $\mathbb{R}$. Rather than state Heine-Borel for $\mathbb{R}$ explicitly at this point, I will wait until Section 3, where I state (and prove) the $\mathbb{R}^N$ version of Heine-Borel.

2 $\mathbb{R}^N$ is complete.

2.1 Convergence and pointwise convergence in $\mathbb{R}^N$.

The proof that $\mathbb{R}^N$ is complete follows almost immediately from the fact that convergence in $\mathbb{R}^N$ is equivalent to *pointwise* convergence, that is, convergence for every coordinate sequence (x_{tn}). Similarly, a sequence (x_t) in $\mathbb{R}^N$ is Cauchy iff all of the coordinate sequences are Cauchy.

To show the equivalence between convergence and pointwise convergence, I begin by recalling the result, proved in the notes on Metric Spaces, that if two metrics are strongly equivalent, then a sequence converges to, say, x^*, under one metric iff it converges under the other, and a sequence is Cauchy under one metric iff it is Cauchy under the other. In $\mathbb{R}^N$, the Euclidean and max metrics are strongly equivalent. This allows me to prove the following.

Theorem 3. *Let (x_t) be a sequence in $\mathbb{R}^N$ (with the Euclidean metric).*

1. *$x_t \to x^*$ iff $x_{tn} \to x^*_n$ for each n,*
2. *(x_t) is Cauchy iff (x_{tn}) is Cauchy for each n.*

Proof. By the the results in the section on Equivalent Metrics in the notes on Metric Spaces, it suffices to check these properties for the $d_{\max}$ metric.

1. $\Rightarrow$. Fix $\varepsilon > 0$. Since $x_t \to_{\max} x^*$, there is a T such that for all $t > T$, $\max_n |x_{tn} - x^*_n| < \varepsilon$, hence $|x_{tn} - x^*_n| < \varepsilon$ for all n, which implies $x_{tn} \to x^*$ for all n.

$\Leftarrow$. Fix $\varepsilon > 0$. Since $x_{tn} \to x^*$ for all n, for each n there is a T_n such that for all $t > T_n$, $|x_{tn} - x^*_n| < \varepsilon$. Take $T = \max\{T_n\}$. Then for all $t > T$, $|x_{tn} - x^*_n| < \varepsilon$ for all n, hence $\max_n |x_{tn} - x^*_n| < \varepsilon$, which implies $x_t \to x^*$.

2. $\Rightarrow$. Fix $\varepsilon > 0$. Since (x_t) is Cauchy under $d_{\max}$, there is a T such that for any $s, t > T$, $d_{\max}(x_s, x_t) < \varepsilon$, hence $|x_{sn} - x_{tn}| < \varepsilon$ for all n, which implies that (x_{tn}) is Cauchy for all n.

   $\Leftarrow$. Fix $\varepsilon > 0$. Since (x_{tn}) is Cauchy for all n, for each n there is a T_n such that for all $s, t > T_n$, $|x_{tn} - x_{sn}| < \varepsilon$. Let $T = \max_n\{T_n\}$. Then for all $s, t > T$, $|x_{tn} - x_{sn}| < \varepsilon$ for all n, hence $d_{\max}(x_t, x_s) < \varepsilon$.

■

Finiteness of N plays a critical role in certain steps in the above arguments. In particular, note that if N were not finite, then the point in the proof where T is set equal to $\max_n\{T_n\}$ could yield $T = \infty$. I return to this issue in the notes on $\mathbb{R}^\omega$.

2.2 $\mathbb{R}^N$ is Complete.

Theorem 4. *$\mathbb{R}^N$ is complete.*

Proof. Let (x_t) be a Cauchy sequence in $\mathbb{R}^N$. Therefore each (x_{tn}) is Cauchy (Theorem 3). Since $\mathbb{R}$ is complete, for each coordinate n there is an x^*_n such that $x_{tn} \to x^*_n$. By Theorem 3, this implies that $x_t \to x^* = (x^*_1, \ldots, x^*_N)$. ■

3 Heine-Borel: Closed and bounded sets in $\mathbb{R}^N$ are compact.

I can now prove the main result about compact sets in $\mathbb{R}^N$.

Theorem 5 (Heine-Borel)**.** *Let $C \subseteq \mathbb{R}^N$. C is compact iff it is closed and bounded.*

Proof. $\Rightarrow$. This is true in any metric space.

$\Leftarrow$. In $\mathbb{R}^N$, any bounded set is totally bounded (see the notes on Metric Spaces). By Theorem 4, $\mathbb{R}^N$ is complete, and any closed subset of a complete set is complete. Therefore any closed and bounded subset of $\mathbb{R}^N$ is complete and totally bounded, and is therefore compact. ■

The following fact is often useful.

Theorem 6. *If $A \subseteq \mathbb{R}^N$ is bounded then $\overline{A}$ is compact.*

Proof. If $A \subseteq \mathbb{R}^N$ is bounded then there is an $r > 0$ such that $A \subseteq N_r(0)$. Then $\overline{A} \subseteq \overline{N_r(0)} = \{x \in \mathbb{R}^N : \|x\| \leq r\}$, and the latter is compact by Theorem 5, since it is closed and it is bounded (by $N_{r+1}(0)$, for example). ■

4 $\mathbb{R}$ is uncountable.

The Heine-Borel theorem implies that $\mathbb{R}$ is uncountable.

Theorem 7. $\mathbb{R}$ *is uncountable.*

Proof. By definition, $\mathbb{R}$ is countable iff there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. Consider, then an arbitrary function $f : \mathbb{N} \to \mathbb{R}$. I show that there is a $y^* \in \mathbb{R}$ such that $y^* \notin f(\mathbb{N})$. Since f was arbitrary, the proof then follows.

For each $t \in \mathbb{N}$, let $x_t = f(t)$. To begin, consider any $y_0 \neq x_0$. Let I_0 be a non-degenerate closed interval containing y_0 but not containing x_0. For example, take $\varepsilon_0 < |x_0 - y_0|/2$ and $I_0 = N_{\varepsilon_0}(y_0)$.

Next choose any $y_1 \in I_0$ such that $y_1 \neq x_1$. Consider any non-degenerate closed interval I_1 such that $I_1 \subseteq I_0$, $y_1 \in I_1$, and $x_1 \notin I_1$.

And so on. We thus have a nested sequence of closed intervals $\dots I_t \subseteq \cdots \subseteq I_1 \subseteq I_0$ with, for each t, $y_t \in I_t$ but $x_t \notin I_t$.

Since all y_t lie in I_0, which is compact by Theorem 5 (Heine-Borel), the sequence (y_t) has a subsequential limit, call it y^*. For each s, all but a finite number of y_t are in I_s, which is closed. Therefore $y^* \in I_t$ for every t. There is, however, no x_t with this property, which implies that $y^* \neq f(t)$ for all t, as was to be shown. ■

5 Generalizations to Abstract Vector Spaces.

Let X be a finite-dimensional vector space with basis Z and consider the associated max norm (and metric) on X. (See the notes on Vectors Spaces and Norms.)

- In the notes on Metric Spaces, the core of the proof that, in $\mathbb{R}^N$, bounded implies totally bounded used the max norm/metric and that core argument extends with essentially no modification to abstract finite-dimensional vector spaces with the max norm.

- Similarly, the core of the proof that $\mathbb{R}^N$ is complete (Theorem 3 and Theorem 4) used the max norm and that same argument shows that an abstract finite-dimensional vector space with the max norm is likewise complete.

Therefore, under the max norm, Heine-Borel holds for any finite-dimensional vector space. In the notes on Existence of Optima, I use this to prove that, in any finite-dimensional vector space, all norms are equivalent. This implies that any finite-dimensional normed vector space, with any norm, is complete and that Heine-Borel holds in any finite-dimensional normed vector space, again with any norm.

In contrast, in the notes on Completeness in $\mathbb{R}^\omega$, I show that Heine-Borel fails catastrophically in *any* infinite-dimensional normed vector space. In particular, in *any* infinite-dimensional normed vector space, *no* closed ball is totally bounded (under the metric induced by the norm), and hence no closed ball is compact.