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Existence of Competitive Equilibrium[1]

Fix an an aggregate endowment $\bar{e} = \sum_i e^i \gg 0$. Let $\bar{f} : \mathbb{R}^N_{++} \to \mathbb{R}^N$ denote aggregate excess demand. As usual, a sufficient condition for $\bar{f}$ to be a function rather than a correspondence is that preferences are strictly convex. The argument below extends to correspondences with a little additional effort.

Let $\partial\mathbb{R}^N_+$ denote the boundary of $\mathbb{R}^N_+$ (which is also the boundary of $\mathbb{R}^N_{++}$): $p \in \partial\mathbb{R}^N_+$ iff $p \in \mathbb{R}^N_+$ and there is at least one coordinate n such that $p_n = 0$.

Definition 1. *An aggregate excess demand function $\bar{f} : \mathbb{R}^N_{++} \to \mathbb{R}^N$ is* standard *iff it satisfies the following five properties.*

1. *$\bar{f}$ is continuous.*
2. *For any $p \in \mathbb{R}^N_{++}$, $p \cdot \bar{f}(p) = 0$.*
3. *For any $p \in \mathbb{R}^N_{++}$ and any $\delta > 0$, $\bar{f}(\delta p) = \bar{f}(p)$.*
4. *For any $p \in \mathbb{R}^N_{++}$, $\bar{f}(p) \geq -\bar{e}$.*
5. *For any sequence $(p_t) \in \mathbb{R}^N_{++}$, if $p_t \to p \in \partial\mathbb{R}^N_+$, $p \neq 0$, then $\|\bar{f}(p_t)\| \to \infty$.*

Property (4) follows from the maintained assumption that the individual consumption space is $\mathbb{R}^N_+$; individual demand cannot (by assumption) be negative, and hence aggregate excess demand for any commodity can never be less than minus the aggregate endowment of that commodity. Sufficient conditions for the other standard properties for excess demand are that all preferences are continuous, strictly convex, and strongly monotone. Property (3) does not play a direct role in the existence theorem but I have included it for completeness.

Properties (2), (4) and (5) together imply a crude form of the Law of Demand, recorded in the next theorem.

Theorem 1. *Consider any sequence $(p_t) \in \mathbb{R}^N_{++}$ such that $p_t \to p \in \partial\mathbb{R}^N_+$, $p \neq 0$. Then the following hold for any standard aggregate excess demand function.*

1. *For any n with $p_n > 0$, $\bar{f}_n(p_t)$ is bounded.*
2. *For at least one n with $p_n = 0$, there is a subsequence (p_{tk}) such that $\bar{f}(p_{tk}) \to \infty$.*

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Proof. By property (4), for any n, $p_{tn}\bar{f}_n(p_t) \geq -p_{tn}\bar{e}_n$. Since $p_t \cdot \bar{f}(p_t) = 0$ (by property (2)), it follows that, for any n,

$$p_{tn}\bar{f}(p_t) = \sum_{\ell \neq n} -p_{t\ell}\bar{f}_\ell(p_t) \leq \sum_{\ell \neq n} p_{t\ell}\bar{e}_\ell \leq p_t \cdot \bar{e}.$$

Hence, since $p_{tn} > 0$,

$$\bar{f}_n(p_t) \leq (p_t \cdot \bar{e})/p_{tn}.$$

If $p_n > 0$, then, since $p_t \to p$, it follows that there is a T such that for all $t > T$,

$$(p_t \cdot \bar{e})/p_{tn} < (p \cdot \bar{e})/p_n + 1,$$

which imples that $\bar{f}_n(p_t)$ is bounded above (by the larger of $(p \cdot \bar{e})/p_n + 1$ and $\max_{t \leq T} \bar{f}_n(p_t)$). Since $\bar{f}_n(p_t)$ is also bounded below (by property 4), $\bar{f}_n(p_t)$ is bounded, which establishes the first part of Theorem 1.

As for the second part of Theorem 1, since $\bar{f}_n(p_t)$ is bounded below (by property (4)) but $\|\bar{f}(p_t)\| \to \infty$ (by property (5)), if follows that there is at least one good n for which there is a subsequence p_{t_k} along which $\bar{f}_n(p_{t_k}) \to \infty$. By the first part of Theorem 1, any such good must have $p_n = 0$. ■

If $\bar{f}_n(p) = 0$ then the total demand and supply for good n are in exact balance. Thus, if $\bar{f}(p) = 0$, then p is a competitive equilibrium price vector. The following result thus establishes existence of competitive equilibrium for excess demand functions that are standard.

Theorem 2. *Suppose that $\bar{f} : \mathbb{R}^N_{++} \to \mathbb{R}^N$ is a standard excess demand function. Then there exists a $p^* \gg 0$ such that $\bar{f}(p^*) = 0$.*

Proof. Let $\Delta = \{p \in \mathbb{R}^N_+ : \sum_n p_n = 1\}$, $\Delta^\circ = \{p \in \Delta : p \gg 0\}$, and $\partial\Delta = \Delta \setminus \Delta^\circ$: $p \in \partial\Delta$ iff $p \in \Delta$ and $p_n = 0$ for at least one n. I will show that there is a $p^* \in \Delta^\circ$ such that $\bar{f}(p^*) = 0$.

The following proof appears in Hildenbrand and Kirman (1988). Define the non-empty-valued correspondence $g : \Delta \to \mathbb{P}(\Delta)$ as follows.

1. For $p \in \Delta^\circ$, define

$$g_n(p) = \frac{p_n + \max\{0, \bar{f}_n(p)\}}{1 + \sum_\ell \max\{0, \bar{f}_\ell(p)\}}.$$

   Thus, g maps Δ° to itself. Note that on Δ°, g is a continuous function, since it is a composition of continuous functions.

2. For $p \in \partial\Delta$, define $g(p)$ to be the *set* given by,

$$g(p) = \{q \in \Delta : p_n > 0 \Rightarrow q_n = 0\}.$$

I claim that g is convex-valued and has a closed graph. Convex-valued is almost immediate from the definition of g and the fact that Δ is convex. As for closed graph, consider any convergent sequence (p_t, q_t) in the graph of g, converging to, say (p, q). Since (p_t, q_t) is a sequence in $\Delta \times \Delta$, which is closed, $(p, q) \in \Delta \times \Delta$. I need to show that $q \in g(p)$.

If $p \in \Delta^\circ$ then $p_t \in \Delta^\circ$ for t large, and the closed graph condition is then immediate from continuity of g on Δ°. Therefore, suppose that $p \in \partial\Delta$. I claim that if $p_n > 0$ then $q_n = 0$, which proves the closed graph claim.

If there is a subsequence (p_{tk}, q_{tk}) for which $p_{tk} \in \partial\Delta$, then for any n for which $p_n > 0$, $p_{tn} > 0$ for all t large, and hence for all such t, $q_{tn} = 0$, hence, since $q_{tn} \to q_n$, $q_n = 0$.

The remaining case is that $p_t \in \Delta^\circ$ for all but at most finitely many t. To simplify notation, suppose that, in fact, $p_t \in \Delta^\circ$ for all t. Theorem 1 then implies that $\sum_n \max\{0, \bar{f}_n(p_t)\} \to \infty$ and also that for any n for which $p_n > 0$, $\bar{f}_n(p_t)$ is bounded. This implies that for any n for which $p_n > 0$, $g_n(p_t) \to 0$. Since $q_{tn} = g_n(p_t)$, and $q_{tn} \to q_n$, it follows that $q_n = 0$, as was to be shown

Since Δ is compact and convex and g is convex-valued and has a closed graph, the Kakutani Fixed Point Theorem applies: there is a fixed point $p^* \in \Delta$ such that $p^* \in g(p^*)$; in fact, $p^* = g(p^*)$. By definition of g, if $p \in \partial\Delta$, then $p \notin g(p)$. Therefore $p^* \in \Delta^\circ$: $p^* \gg 0$, as required. By definition of g, for any n,

$$p_n^* \left(1 + \sum_\ell \max\{0, \bar{f}_\ell(p^*)\}\right) = p_n^* + \max\{0, \bar{f}_n(p^*)\}$$

or

$$p_n^* \left(\sum_\ell \max\{0, \bar{f}_\ell(p^*)\}\right) = \max\{0, \bar{f}_n(p^*)\},$$

hence, multiplying both sides by $\bar{f}_n(p^*)$

$$p_n^* \bar{f}_n(p^*) \left(\sum_\ell \max\{0, \bar{f}_\ell(p^*)\}\right) = \bar{f}_n(p^*) \max\{0, \bar{f}_n(p^*)\}.$$

Adding across n and applying Walras's Law, $p^* \cdot \bar{f}(p^*) = 0$, then yields

$$\sum_n \bar{f}_n(p^*) \max\{0, \bar{f}_n(p^*)\} = 0.$$

This expression only holds if for every n,

$$\bar{f}_n(p^*) \leq 0.$$

$p^* \cdot \bar{f}(p^*) = 0$, and $p^* \gg 0$, then implies $\bar{f}(p^*) = 0$. ■

References

Hildenbrand, W. and A. P. Kirman. 1988. *Equilibrium Analysis.* Amsterdam: North-Holland.