John Nachbar
Washington University in St. Louis
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Continuity and Connectedness.[1]

Theorem 1. *Let (X, d_x) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be continuous. If X is connected then $f(X)$ is connected.*

Proof. I argue by contraposition. Suppose that $f(X)$ is not connected. Then there are open, disjoint, non-empty sets $V_1, V_2 \subseteq f(X)$ (open in the $f(X)$ metric space) such that $V_1 \cup V_2 = f(X)$. Let $O_1 = f^{-1}(V_1)$ and $O_2 = f^{-1}(V_2)$. By continuity, O_1 and O_2 are open. Then O_1, O_2 are disjoint, since V_1, V_2 are disjoint. And, $X = O_1 \cup O_2$, since $f(X) = V_1 \cup V_2$. Finally, O_1, O_2 are not empty, since V_1, V_2 are not empty. ■

The above theorem says that connectedness is a topological property, meaning a property that is preserved by transformation by a continuous function. Compactness is another important example of a topological property. In contrast, convexity is *not* a topological property.

Theorem 2 (Intermediate Value Theorem)**.** *Let $[a, b] \subseteq \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ then, for any $y \in (f(a), f(b))$, there is an $x \in (a, b)$ such that $f(x) = y$. And an analogous result holds if $f(b) < f(a)$.*

Proof. $f([a, b])$ is an interval, since it is connected and since any connected set in $\mathbb{R}$ is an interval. Suppose $f(a) < f(b)$. Since $f(a), f(b) \in f([a, b])$, it follows that $[f(a), f(b)] \subseteq f([a, b])$. So, if $y \in (f(a), f(b))$ then there is an $x \in [a, b]$ such that $f(x) = y$. Since $y \neq f(a)$ and $y \neq f(b)$, $x \in (a, b)$. The argument for $f(b) < f(a)$ is similar. ■

Example 1. Suppose $f(a) < 0$ and $f(b) > 0$. Then there is an $x \in (a, b)$ such that $f(x) = 0$.

This fact can be used to show the existence of a competitive equilibrium if there are only 2 commodities. To handle the case of more than 2 commodities, more sophisticated machinery must be employed (in particular, the Brouwer fixed point theorem). □