Optimal portfolios and admissible strategies in a Lévy market

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Stochastic Analysis at Purdue Workshop 2009

Outline

Merton's portfolio optimization problem

1. Set-up: A frictionless market consisting of a risky asset with price process $S_t : \Omega \to \mathbb{R}_+$, $t \geq 0$, and a risk-free asset with price process B_t, $t \geq 0$, s.t. $B_0 = 1$.
2. Goal:

   Allocate a given initial wealth w_0 so that to maximize the agent's expected final "utility" during a finite time horizon $[0, T]$.
3. State-dependent utility: $U(w, \omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ s.t.
   - Increasing and concave in the wealth w, for each state of nature $\omega \in \Omega$.
   - Differentiable in $[0, H(\omega))$ and flat for wealths w above certain threshold $H(\omega) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$.
4. Problem:

   Find a *self-financing* portfolio strategy such that the corresponding portfolio's value process $\{V_t\}_{t \leq T}$ maximizes $\mathbb{E}\{U(V_T, \omega)\}$ subject to $V_0 \leq w$ (budget constraint) and $V_\cdot \geq 0$ (solvency or admissibility condition).

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Some financial background

1. Typical set-up:
   - The discounted price process $\{B_t^{-1} S_t\}_{t \geq 0}$ is a *semimartingale*.
   - The class $\mathcal{M}$ of Equivalent Martingale Measures (EMM) is non-empty.
2. The Fundamental Theorem of Finance:
   - The market is complete if and only if $\mathcal{M} = \{\mathbb{Q}\}$.
   - “Any" T-claim H is reachable with the initial endowment $w = \mathbb{E}_{\mathbb{Q}} \left\{ B_T^{-1} H \right\}$.
   - Each EMM Q induces an arbitrage-free pricing procedure: $\mathbb{E}_{\mathbb{Q}} \left\{ B_T^{-1} H \right\}$.
3. The Super-Replication Theorem: [Kramkov 97]

   The cost of super-replication is $\bar{w} := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} H \right]$
4. There exists an admissible portfolio $\{V_t\}_{t \leq T}$ such that

   $$V_0 = w \quad \text{and} \quad V_T \geq H, \quad a.s.$$

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Convex Duality Method

- Basic idea: Upper bound a maximization problem with constraints, using a convex minimization problem without constraints.
  - Primal problem:
    $$p^* := \max f(x)$$
    $$s.t. \quad h(x) \leq 0$$
  - Construction of the dual problem:
    - $f(x) \leq \overbrace{f(x) - \lambda h(x)}^{\mathcal{L}(x,\lambda)}, \quad \lambda \geq 0$
    - $p^* \leq \widetilde{\mathcal{L}}(\lambda) := \max_x \mathcal{L}(x, \lambda),$ Convex
    - $p^* \leq \underbrace{d^* := \min_{\lambda \geq 0} \widetilde{\mathcal{L}}(\lambda)}_{\text{Dual Problem}}.$
- We say that strong duality holds if $p^* = d^*$.

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Convex duality in portfolio optimization problems

Karatzas et. al. 91, Cvitanić & Karatzas 92-93, Kramkov & Schachermayer 99

1. The primal problem:

$$\begin{cases} p^*(w) := \sup \mathbb{E}\left\{U(V_T, \omega)\right\} \\ \text{such that} \quad V_0 \leq w \quad \text{and} \quad V_\cdot \geq 0, \end{cases}$$

2. Assumption: $w < \bar{w} := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} H\right] < \infty$.
3. The dual domain $\widetilde{\Gamma}$:

   Nonnegative supermartingales $\{\xi_t\}_{t \geq 0}$ such that (i) $0 \leq \xi_0 \leq 1$ and (ii) $\{\xi_t B_t^{-1} V_t\}_{t \geq 0}$ is a supermaringale for all admissible $\{V_t\}_{t \geq 0}$.
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Motivation behind the dual problem

- $\mathbb{E}\ \{\xi_T B_T^{-1} V_T\} \le \xi_0 B_0^{-1} V_0 \le w \quad$ if $V_0 \le w$.
- For any $\xi \in \widetilde{\Gamma}$, $\lambda > 0$, and admissible $V.$ with $V_0 \le w$:

$$\begin{aligned}
\mathbb{E}\{U(V_T,\omega)\} &\le \mathbb{E}\{U(V_T,\omega)\} - \lambda\left(\mathbb{E}\{\xi_T B_T^{-1} V_T\} - w\right) \\
&= \mathbb{E}\{U(V_T,\omega) - \lambda\xi_T B_T^{-1} V_T\} + \lambda w \\
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- For any subclass of $\Gamma \subset \widetilde{\Gamma}$:

$$p^*(w) = \sup \mathbb{E}\{U(V_T,\omega)\} \le \underbrace{\inf_{\xi\in\Gamma} \mathbb{E}\left\{\widetilde{U}(\lambda\xi_T B_T^{-1},\omega)\right\}}_{d_\Gamma^*(\lambda)} + \lambda w.$$

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Relationship between the dual and primal problems

The Dual Theorem. [KrSch 99, FllmLkrt, 2000]

1. Weak duality:

$$p^*(w) \le d^*_{\Gamma}(\lambda) + \lambda w, \quad \text{for all } \lambda > 0, \text{ and } \Gamma \subset \widetilde{\Gamma}.$$

2. Strong duality:

$$p^*(w) = d^*_{\widetilde{\Gamma}}(\lambda^*) + \lambda^* w, \quad \text{for some } \lambda^* > 0.$$

3. Dual characterization of the optimal final wealth:
   The primal problem is attainable at an admissible portfolio V^* s.t.

$$V^*_T = I\left(\lambda^* \, \xi^*_T \, B_T^{-1}\right),$$

   where $I(\cdot, \omega)$ is the "inverse" of $U'(\cdot, \omega)$, and λ^* is the dual solution of $d^*_{\widetilde{\Gamma}}$.

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The Dual Theorem. [KrSch 99, FllmLkrt, 2000]

1. Weak duality:

$$p^*(w) \leq d^*_{\Gamma}(\lambda) + \lambda w, \quad \text{for all } \lambda > 0, \text{ and } \Gamma \subset \widetilde{\Gamma}.$$

2. Strong duality:

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3. Dual characterization of the optimal final wealth:
   The primal problem is attainable at an admissible portfolio V^* s.t.

$$V^*_T = I\left(\lambda^* \, \xi^*_T \, B_T^{-1}\right),$$

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A non-Markovian Lévy Market

1. Let W be a Wiener process and let N be an independent Poisson jump measure associated with a Lévy process Z with Lévy measure ν:

$$N((0,\tau] \times (a,b]) := \#\{t \leq \tau : \Delta Z_t \in (a,b]\} \sim \text{Poisson}(\tau\nu((a,b])).$$

2. The stock price process $\{S_t\}_{t\geq 0}$ follows the dynamics:

$$d\,S_t = S_{t^-}\left\{\mu_t\,dt + \sigma_t\,dW_t + \int_{\mathbb{R}^d} v(t,z)\,(N(dt,dz) - dt\nu(dz))\right\},$$

where $v(t,0) = 0$, and $v(t,z) > -1$.

3. Interpretation: (Finite-jump activity $\nu(\mathbb{R}) < \infty$)
    - Between jump times the stock follows a Black-Scholes model with instantaneous mean rate of return $\mu_t - \int v(t,z)\nu(dz)$ and volatility σ_t.
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Merton's problem in Lévy markets

A natural problem:

For a specific market model (say Lévy one) and a given utility function,

Can one narrow down the dual domain $\Gamma \subset \widetilde{\Gamma}$ *where to search* ξ^* *?*

Theorem. [Karatzas et. al. 91], [Kunita, 03]

For the previous Lévy market and for unbounded Inada type utility functions, the Dual Theorem holds and the dual solution ξ^* is the *stochastic exponential* $\mathcal{E}(X^*)$ of a local martingale

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Key tool

Representation Theorems: [Kunita-Watanabe (1967)]

Let $\mathcal{F}_t$ be the information process generated by $\{W_s : s \leq t\}$ and by $\{Z_s : s \leq t\}$.

- ξ is a *positive* local martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ iff $\xi_t = \xi_0 \mathcal{E}(X)$ with

$$X_t := \int_0^t G(s) dW_s + \int_0^t \int F(s,z) \left(N(dt,dz) - dt\nu(dz) \right), \quad F > -1.$$

- ξ is a *positive* supermartingale iff $\xi_t = \xi_0 \mathcal{E}(X - A)$ where X is as above and A is increasing predictable s.t. the jump $\Delta A < 1$.

A closer look into the dual theorem

1. WLG assume $B_t \equiv 1$. Let Γ be a convex subclass of $\widetilde{\Gamma}$ s.t.
   - (i) $\bar{w}_\Gamma := \sup_{\xi \in \Gamma} \mathbb{E}\{\xi_T H\} < \infty$
   - (ii) Γ is closed under "Fatou convergence".

   Then, for each $0 < w < \bar{w}_\Gamma$, there exist $\lambda^* > 0$ and $\xi^* \in \Gamma$ s.t.
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2. Furthermore, if
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F-L & Ma, 2008

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fulfills the conditions necessary (i)-(iii) for the *Dual Theorem*.

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is super-replicable by an admissible portfolio V^* with $V_0^* \leq w$ and

$$\mathbb{E}U(V_T) \leq \mathbb{E}U(V_T^*), \quad \forall \quad V \quad s.t. \quad V_0 \leq w.$$

Construction of the dual class Γ in Lévy markets

F-L & Ma, 2008

1. Let

$$\mathcal{S} := \{X_t := \int_0^t G(s)dW_s + \int_0^t \int F(s,z)\widetilde{N}(ds,dz) : F \geq -1\},$$

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Characterization of the dual class

Question: Under what conditions $\xi = \xi_0 \mathcal{E}(X - A) \in \widehat{\Gamma}$ is in $\widetilde{\Gamma}$?

1. There exists predictable increasing A^p s.t. $\xi = \xi_0 \mathcal{E}(X - A^p)$.
2. $\{\xi_t S_t\}_t$ is a supermartingale iff

$$h_t := b_t + \sigma_t G(t) + \int v(t,z) F(t,z) \nu(dz) \le a_t := \frac{dA^c_t}{dt}.$$

   for any $t \le \tau(\omega) := \sup_n \inf\{t : \xi_t < 1/n\}$.
3. $\xi_. V_.^\beta$ is supermartingale for any admissible $V_t^\beta := V_0 + \int_0^t \beta_u \frac{dS_u}{S_u}$ iff

$$h_t \beta_t \le a_t, \quad a.e. \quad t \le \tau.$$

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An example

Model: ν is atomic with atoms $\{z_i\}_i$:

$$d\,S_t = S_{t-}\left\{\mu_t\,dt + \sigma_t\,dW_t + \sum_i v(t,z_i)(t)dN_t^{(i)}\right\},$$

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- Such a parametrization could potentially lead to numerical approximation schemes of the solution.
- The approach can be applied to more general jump-diffusion models driven by Lévy processes such as

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For Further Reading I

Figueroa-Lopez and Ma.
State-dependent utility maximization in Lévy markets
Preprint available at ArXiv, 2008.

Kramkov and Schachermayer.
The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Finance and Stochastics*, 1999.

Föllmer and Leukert.
Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 2000.

Karatzas, Lehoczky, Shreve, and Xu.
Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control and Optimization*, 1991.

For Further Reading II

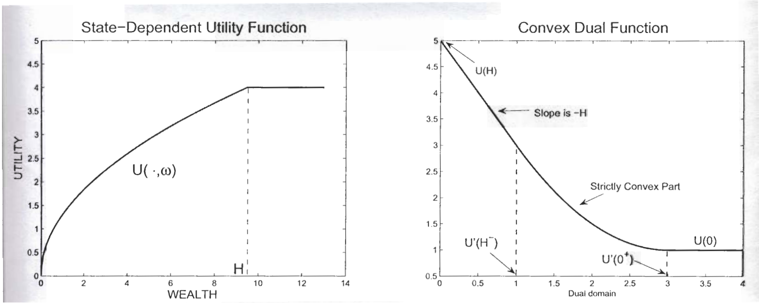
Kunita

Variational equality and portfolio optimization for price processes with jumps. *In Stoch. Proc. and Appl. to Mathem. Fin.*, 2003.

Kramkov.

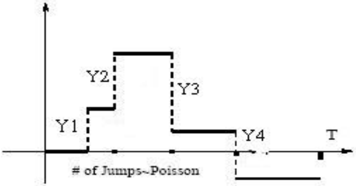
Optional decomposition of supermartingales and pricing of contigent claims in incomplete security markets. *Prob. Th. and Rel. fields*, 1996.

Utility function and its convex dual function

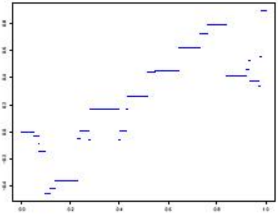


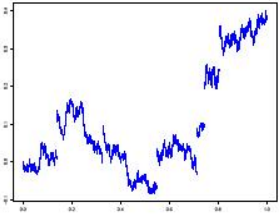
- $\widetilde{U}(\lambda, \omega) := \sup_{0 \le w \le H} \{U(w, \omega) - \lambda\, w\},$
- $I(\lambda) := \inf\{w \ge 0 : U'(w) \le \lambda\} = -\widetilde{U}'(\lambda).$

Lévy processes with jumps



Compound Poisson Process





Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion