Small-time expansions for local jump-diffusions models with infinite jump activity

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Outline

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Setup

1. $(\Omega, \mathcal{F}, \mathbb{P})$: complete probability space;
2. (W_t): Wiener process (i.e. continuous process with independent and stationary increments such that $W_t \sim \mathcal{N}(0, t)$);
3. (Z_t): Lévy process (i.e. cádlág independent and stationary increments s.t. $Z_0 = 0$) without “Gaussian component" and with “Lévy density" $h : \mathbb{R}\backslash\{0\} \to \mathbb{R}_+$:

$$\mathbb{E}\exp(iuZ_t) = \exp\left\{ t\left(iub + \int_{\mathbb{R}\backslash\{0\}} (e^{iuz} - 1 - iuz\mathbf{1}_{\{|z|\leq 1\}})h(z)dz \right) \right\}.$$

4. W and Z are independent;

Local jump-diffusion model

1. Assume Z to be of *bounded variation* without drift (i.e. $Z_t = \sum_{s \le t} \Delta Z_s$);
2. Fix deterministic functions:

$$b(x) : \mathbb{R} \to \mathbb{R}, \quad \sigma(x) : \mathbb{R} \to \mathbb{R}, \quad \gamma(x,z) : \mathbb{R} \times \mathbb{R} \backslash \{0\} \to \mathbb{R}, \quad s.t. \quad \gamma(x,0) = 0;$$

3. The Model:

$$X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s + \sum_{s \le t} \gamma(X_{s-}^x, \Delta Z_s).$$

4. Existence and uniqueness of (X_t^x) are guaranteed under standard linear growth and Lipschitz conditions on the coefficients b, σ, and γ;
5. More general process:

$$X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s + \int_0^t \int_{|z| \le 1} \gamma(X_{s-}^x, z) \bar{\mu}_Z(ds, dz),$$

where $\bar{\mu}_Z := \mu_Z - \mathbb{E}\mu_Z$ is the compensated jump measure of Z:

$$\mu_Z((u,v] \times [c,d]) := \# \{ t \in (u,v] : \Delta Z_t \in [c,d] \}, \qquad (0 < u < v, c < d).$$

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Properties

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2. (X_t^x) is a Homogeneous Markov Process:
$$\mathbb{P}\left(X_T^x \in [c,d] \middle| X_s^x, s \leq t\right) = \mathbb{P}\left(X_T^x \in [c,d] \middle| X_t^x\right), \qquad (t \leq T);$$
$$\mathbb{P}\left(X_T^x \in [c,d] \middle| X_t^x = y\right) = \mathbb{P}\left(X_{T+h}^x \in [c,d] \middle| X_{t+h}^x = y\right), \qquad (t \leq T; h > 0).$$
3. Dynkin's Formula:
$$\mathbb{E}\left[f(X_t^x)\right] = f(x) + \int_0^t \mathbb{E}[Lf(X_s^x)]ds,$$
for $f \in C_b^2$, where L is the so-called Generator Operator of X:
$$(Lf)(y) = b(x)f'(y) + \frac{\sigma^2(y)}{2}f''(y) + \int_{\mathbb{R}\setminus\{0\}} (f(y+\gamma(y,z)) - f(y))h(z)dz.$$
4. In particular, $Lf(x) = \lim_{t\to 0} \frac{\mathbb{E}[f(X_t^x)]-f(x)}{t} \iff \mathbb{E}[f(X_t^x)] = f(x) + tLf(x) + o(t)$;
5. If $f, Lf, \ldots, L^n f := L(L^{n-1}f)$ are C_2^b, then
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Short-time asymptotics for the transition distributions

1. Objective 1

   Fix $x \in \mathbb{R}$ and $y > 0$. Study the rate of convergence of $\mathbb{P}(X_t^x \geq x + y)$ to 0 as $t \to 0$:

$$\mathbb{P}(X_t^x \geq x + y) = A_1(y; x)t + A_2(y; x)\frac{t^2}{2} + \cdots + A_n(y; x)\frac{t^n}{n!} + o(t^{n+1}).$$

2. Objective 2

   Study the asymptotic behavior of transition density $p_t(z; x) = \frac{d}{dz}\mathbb{P}(X_t^x \leq z)$ (when it exists) in short-time;

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Applications

1. Non-parametric estimation methods for σ, γ, b based on high-frequency observations $X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}$ of X (i.e. when time-span Δ);

$$\frac{1}{\#\{X_{k\Delta} \in (x-\delta, x+\delta)\}} \sum_{X_{k\Delta} \in (x-\delta, x+\delta)} \mathbf{1}_{\{X_{(k+1)\Delta} \geq y+x\}} \longrightarrow A_1(y; x|\gamma, h)\Delta + o(\Delta).$$

2. Numerical valuation of moments $\mathbb{E}[\Phi(X_T^x)]$:
Let $F_t(x;\Phi) := \mathbb{E}[\Phi(X_t^x)]$ and $\widetilde{F}_t(x;\Phi)$ be an "explicit" approximation in short-time;
Then, for $m \in \mathbb{N}$ and $\Delta := T/m$,

$$F_T(x;\Phi) = \mathbb{E}[\mathbb{E}[\Phi(X_T^x)|X_\Delta^x]] = \mathbb{E}[F_{T-\Delta}(X_\Delta^x;\Phi)] \approx \widetilde{F}_\Delta(x;\widetilde{\Phi}),$$

where $\widetilde{\Phi}(y) := F_{T-\Delta}(y;\Phi)$, which in turn is approximated as before, replacing T by $T-\Delta$. Convergence is attained whenever $\tilde{F}_t$ is a second-order approximation.

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Literature

- Léandre (1987):
  For $z \neq x$ and $\sigma \equiv 0$ (Pure-jump case),
  $$\lim_{t\to 0}\frac{1}{t}p_t(z;x) = -\frac{d}{dz}\int_{\{\zeta:\gamma(x,\zeta)\geq z\}} h(\zeta)d\zeta =: g(z-x;x) \qquad \text{(“Lévy density" of } (X_t^x))$$
  In particular, we expect, for $y > 0$,
  $$\lim_{t\to 0}\frac{1}{t}\mathbb{P}(X_t^x \geq x+y) = \lim_{t\to 0}\frac{1}{t}\int_{x+y}^{\infty} p_t(z;x)dz = \int_y^{\infty} g(u;x)du.$$
- Ishikawa (1994) and Picard (1996):
  Let $z > x$. Note that
  $$\mathrm{supp}(h)\cap\{\zeta:\gamma(x,\zeta)\geq z\} = \emptyset \quad \Longrightarrow \quad \lim_{t\to 0}\frac{1}{t}p_t(z;x) = 0.$$
  In that case, $p_t(z;x) \sim t^{\kappa} g_{\kappa}(z;x)$, where κ is the minimal number of jumps needed to reach z from x and $g_{\kappa}(z;x) \neq 0$.

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  In that case, $p_t(z; x) \sim t^{\kappa} g_{\kappa}(z; x)$, where κ is the minimal number of jumps needed to reach z from x and $g_{\kappa}(z; x) \neq 0$.

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  For a "general Lévy process" (X_t),

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  1. Provide a formal proof of expansions (1)-(2);
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Small-time expansions for Lévy tail distributions

1. Key Assumption: For all $\varepsilon > 0$ and $k \geq 0$, the Lévy density h of X is smooth s.t.

$$\sup_{|x| \geq \varepsilon} |h^{(k)}(x)| < \infty;$$

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Outline

Assumptions

1. Z has a $C^{\infty}(\mathbb{R}\backslash\{0\})$ non-zero Lévy density h such that, for every $\varepsilon > 0$ and $n \geq 0$,

$$\text{(i)} \int (1 \wedge |\zeta|) h(\zeta) d\zeta < \infty, \quad \text{and} \quad \text{(ii)} \sup_{|\zeta| > \varepsilon} |h^{(n)}(\zeta)| < \infty.$$

2. For any $x, \zeta \in \mathbb{R}$, $\gamma(x, 0) = 0$, $\gamma(\cdot, \cdot) \in C^{\infty}$, and

$$\sup_{x} \left| \frac{\partial^{i} \gamma(x,\zeta)}{\partial x^{i}} \right| \leq C_i(\zeta^2 \wedge 1), \quad \left| 1 + \frac{\partial \gamma(x,\zeta)}{\partial x} \right| \geq \delta, \qquad \text{for some } C_i < \infty, \delta > 0.$$

3. For fix $\varepsilon > 0$, let $\phi_{\varepsilon} \in C^{\infty}(\mathbb{R})$ s.t. $\mathbf{1}_{|\zeta| \geq \varepsilon} \leq \phi_{\varepsilon}(\zeta) \leq \mathbf{1}_{|\zeta| \geq \varepsilon/2}$. Let

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Then, for some $\varepsilon_0 > 0$ and all $\varepsilon < \varepsilon_0$,

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Second-order expansion for the transition tail distributions

Theorem (F-L & Ouyang, 2011)

For any $x \in \mathbb{R}$ and $y > 0$,

$$\mathbb{P}(X_t^x \geq x + y) = tA_1(x;y) + \frac{t^2}{2}A_2(x;y) + o(t^2), \quad as \quad t \to 0,$$

where $A_1(x;y)$ admits the representation:

$$A_1(x;y) := \int_{\{\gamma(x,\zeta)\geq y\}} h(\zeta)d\zeta.$$

Remarks:

1. Leading term depends only on the jump component;
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Let $h_\varepsilon(\zeta) = \phi_\varepsilon(\zeta)h(\zeta)$ *and* $\bar{h}_\varepsilon(\zeta) = (1-\phi_\varepsilon(\zeta))h(\zeta)$. *Then, there exists a* $\varepsilon_0 > 0$ *small-enough such that for all* $0 < \varepsilon < \varepsilon_0$, A_2 *admit the following representations:*

$$A_2(x;y) := \mathcal{D}(x;y) + \mathcal{J}_1(x;y) + \mathcal{J}_2(x;y),$$

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$$\mathcal{J}_1(x;y) = \int\int \left(\mathbf{1}_{\{\gamma(x+\gamma(x,\bar{\zeta}),\zeta)+\gamma(x,\bar{\zeta})\geq y\}} - \mathbf{1}_{\{\gamma(x,\zeta)\geq y\}}\right) h_\varepsilon(\zeta)d\zeta \bar{h}_\varepsilon(\bar{\zeta})d\bar{\zeta} + \int\int \left(\mathbf{1}_{\{\gamma(x,\zeta)+\gamma(x,\bar{\zeta})\geq y\}} - \mathbf{1}_{\{\gamma(x,\zeta)\geq y\}}\right) h_\varepsilon(\zeta)d\zeta \bar{h}_\varepsilon(\bar{\zeta})d\bar{\zeta},$$

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\mathcal{J}_2(x;y) &= \int\int \mathbf{1}_{\{\gamma(x+\gamma(x,\zeta_1),\zeta_2)+\gamma(x,\zeta_1)\geq y\}} h_\varepsilon(\zeta_1)h_\varepsilon(\zeta_2)d\zeta_1 d\zeta_2 \\
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Theorem (F-L & Ouyang, 2011)

Let $h_\varepsilon(\zeta) = \phi_\varepsilon(\zeta)h(\zeta)$ *and* $\bar{h}_\varepsilon(\zeta) = (1 - \phi_\varepsilon(\zeta))h(\zeta)$. *Then, there exists a* $\varepsilon_0 > 0$ *small-enough such that for all* $0 < \varepsilon < \varepsilon_0$, A_2 *admit the following representations:*

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where

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1. State-free jumps (i.e. $\gamma(x;\zeta) := \gamma(\zeta)$):

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$$\begin{aligned} A_2(x;y) &= 2b(x)g(y) - \sigma^2(x)g'(y) \\ &+ 2\int\int \left(\mathbf{1}_{\{\gamma(\zeta)+\gamma(\bar\zeta)\geq y\}} - \mathbf{1}_{\{\gamma(\zeta)\geq y\}}\right) h_\varepsilon(\zeta)d\zeta \bar h_\varepsilon(\bar\zeta)d\bar\zeta \\ &+ \int\int \left(\mathbf{1}_{\{\gamma(\zeta_2)+\gamma(\zeta_1)\geq y\}} - 2\mathbf{1}_{\{\gamma(\zeta_1)\geq y\}}\right) h_\varepsilon(\zeta_1)h_\varepsilon(\zeta_2)d\zeta_1 d\zeta_2 \end{aligned}$$

2. - $b(x) > 0$ increases the tail probability $\mathbb{P}(X_t^x \geq x+y)$ by $b(x)g(y)$.
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Second-order expansion for the transition densities

Theorem (F-L & Ouyang, 2011)

In addition to the previous conditions assume that $b(x), \sigma^2(x) \in C_b^\infty$ *and* $\sigma(x) \geq \delta$ *for some* $\delta > 0$*. Then,*

$$p_t(x+y;x) = t a_1(x;y) + \frac{t^2}{2} a_2(x;y) + O(t^3), \quad as \quad t \to 0. \tag{3}$$

There exists a $\varepsilon_0 > 0$ *small-enough such that for all* $0 < \varepsilon < \varepsilon_0$*,* A_1 *and* A_2 *admit the following representations:*

$$a_1(x;y) := -\frac{\partial}{\partial y} A_1(x;y), \qquad a_2(x;y) := -\frac{\partial}{\partial y} A_2(x;y).$$

Outline

Some needed notation

1. $Z_t(\varepsilon) = \sum_{s\leq t} \phi_\varepsilon(\Delta Z_s)\Delta Z_s$ ("big" jumps) and $Z'_t(\varepsilon) = \sum_{s\leq t}(1-\phi_\varepsilon(\Delta Z_s))\Delta Z_s$ ("small jumps").
2. $Z_t(\varepsilon) = \sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon$, with (N_t^ε) Poisson with intensity $\lambda_\varepsilon := \int \phi_\varepsilon(z)h(z)dz$ and $J_i^\varepsilon \overset{\text{i.i.d.}}{\sim} h_\varepsilon(z)/\lambda_\varepsilon$.
3. Define the following processes for a set $\pi := \{s_1, \dots, s_n\} \subset \mathbb{R}_+$:

$$X_t(\varepsilon, \emptyset, x) := x + \int_0^t b(X_u(\varepsilon,\emptyset,x))du + \int_0^t \sigma(X_u(\varepsilon,\emptyset,x))\,dW_u + \sum_{0<u\leq t} \gamma\left(X_{u^-}(\varepsilon,\emptyset,x), \Delta Z'_u(\varepsilon)\right);$$

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2. $Z_t(\varepsilon) = \sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon$, with (N_t^ε) Poisson with intensity $\lambda_\varepsilon := \int \phi_\varepsilon(z)h(z)dz$ and $J_i^\varepsilon \overset{\text{i.i.d.}}{\sim} h_\varepsilon(z)/\lambda_\varepsilon$.
3. Define the following processes for a set $\pi := \{s_1, \dots, s_n\} \subset \mathbb{R}_+$:

$$X_t(\varepsilon, \emptyset, x) := x + \int_0^t b(X_u(\varepsilon,\emptyset,x))du + \int_0^t \sigma\left(X_u(\varepsilon,\emptyset,x)\right) dW_u + \sum_{0<u\leq t} \gamma\left(X_{u-}(\varepsilon,\emptyset,x), \Delta Z'_u(\varepsilon)\right);$$

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4. $L_\varepsilon f(y) := \frac{\sigma^2(y)}{2} f''(y) + b(y)f'(y) + \int \left(f(y+\gamma(y,\zeta)) - f(y)\right) \bar{h}_\varepsilon(\zeta)d\zeta.$

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Key steps

1. Condition on the number of “big jumps" by time t:

$$\mathbb{P}(X_t^x \geq x + y) = e^{-\lambda_\varepsilon t} \sum_{n=0}^{\infty} \mathbb{P}\left(X_t^x \geq x + y \,\middle|\, N_t^\varepsilon = n\right) \frac{(\lambda_\varepsilon t)^n}{n!}.$$

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3. Tail control when there is no big jumps:

$$\mathbb{P}\left(|X_t(\varepsilon, \emptyset, x) - x| \geq y\right) \leq Ct^N,$$

for all $t \leq 1$ and $\varepsilon < \varepsilon_0$, and some small-enough $\varepsilon_0 = \varepsilon_0(N, y)$ and $C = C(\varepsilon) < \infty$.

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$$H_0(z) = \mathbb{E}[H_1(X_{t-s}(\varepsilon, \emptyset, z))] = \underbrace{H_1(z; x, y, z)}_{H_{1,0}(z;x,y)} + \underbrace{L_\varepsilon H_1(z; x, y, z)}_{H_{1,1}(z;x,y)}(t-s) + O(t^2);$$

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For Further Reading I

Figueroa-López & Ouyang.
Small-time expansions for local jump-diffusions models with infinite jump activity
Preprint, 2011. Available at www.stat.purdue.edu/∼figueroa.

Figueroa-Lopez & Houdré.
Small-time expansions for the transition distributions of Lévy processes.
Stochastic Processes and Their Applications, 119:3862-3889, 2009.

Rüschendorf & Woerner's Approach

1. Consider the Lévy-Itô decomposition:

$$X_t = \underbrace{X_t^{cp,\varepsilon}}_{\text{Comp. Poiss. of jumps} > \varepsilon} + \underbrace{b_\varepsilon t + \sigma B_t + \lim_{\delta \searrow 0} \left(X_t^{(\delta,\varepsilon)} - \mathbb{E} X_t^{(\delta,\varepsilon)} \right)}_{\bar{X}_t^{(\varepsilon)} := \text{Drift + Brownian + Small-jumps}}.$$

2. Conditioning on the number of jumps of $X^{cp,\varepsilon}$ by time t,

$$\mathbb{P}(X_t \geq x) = e^{-\lambda_\varepsilon t} \sum_{k=0}^{n} \frac{(\lambda_\varepsilon t)^k}{k!} + O(t^{n+1});$$

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