Estimation of NIG and VG models for high frequency financial data

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Abstract: Numerous empirical studies have shown that certain exponential Lévy models are able to fit the empirical distribution of daily financial returns quite well. By contrast, very few papers have considered intraday data in spite of their growing importance. In this paper, we fill this gap by studying the ability of the Normal Inverse Gaussian (NIG) and the Variance Gamma (VG) models to fit the statistical features of intraday data at different sampling frequencies. We propose to assess the suitability of the model by analyzing the signature plots of the point estimates at different sampling frequencies. Using high frequency transaction data from the U.S. equity market, we find the estimator of the volatility parameter to be quite stable at a wide range of intraday frequencies, in sharp contrast to the estimator of the kurtosis parameter, which is more sensitive to market microstructure effects. As a secondary contribution, we also assess the performance of the two most favored parametric estimation methods, the Method of Moments Estimators (MME) and the Maximum Likelihood Estimators (MLE), when dealing with high frequency observations. By Monte Carlo simulations, we show that neither high frequency sampling nor maximum likelihood estimation significantly reduces the estimation error of the volatility parameter of the model. On the contrary, the estimation error of the parameter controlling the kurtosis of log returns can be significantly reduced by using MLE and high-frequency sampling. Both of these results appear to be new in the literature on statistical analysis of high frequency data.

Keywords and phrases: Exponential or geometric Lévy models, High-frequency based estimation, Normal Inverse Gaussian model, Maximum likelihood estimation, Method of moment estimation, Variance Gamma model.

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1. Introduction

Driven by the necessity to incorporate the observed stylized features of asset prices, continuous-time stochastic modeling has taken a predominant role in the financial literature over the past two decades. Most of the proposed models are particular cases of a stochastic volatility component driven by a Wiener process superposed with a pure-jump component accounting for the discrete arrival of major influential information. Accurate approximation of the complex phenomenon of trading is certainly attained with such a general model. However, accuracy comes with a high cost in the form of hard estimation and implementation issues as well as over-parameterized models. In practice and certainly for the purpose motivating the task of modeling in the first place, a parsimonious model with relatively few parameters is desirable. With this motivation in mind, *parametric* Exponential Lévy Models (ELM) are one of the most tractable and successful alternatives to both stochastic volatility models and more general Itô semimartingale models with jumps.

The literature of geometric Lévy models is quite extensive (see Cont & Tankov (2004) for a review). Due to their appealing interpretation and tractability, in this work we concentrate on two of the most popular classes: the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) models proposed by Carr *et al.* (1998) and Barndorff-Nielsen (1998), respectively. In the "symmetric case" (which is a reasonable assumption for equity prices), both models require only one additional parameter, κ, compared to the two-parameter geometric Brownian motion (also called the Black-Scholes model). This additional parameter can be interpreted as the percentage excess kurtosis relative to the normal distribution and, hence, this parameter is mainly in charge of the tail thickness of the log return distribution. In other words, this parameter will determine the frequency of "excessively" large positive or negative returns. Both models are pure jump models with *infinite jump activity*[1]. Nevertheless, one of the parameters, denoted by σ, controls the variability of the log returns and, thus, it can be interpreted as the volatility of the price process.

Numerous empirical studies have shown that certain parametric exponential Lévy models, including the VG and the NIG models, are able to fit *daily returns* extremely well using standard estimation methods such as maximum likelihood estimators (MLE) or method of moment estimators (MME) (c.f. Eberlein & Keller (1995), Eberlein & Özkan (2003), Carr *et al.* (1998), Barndorff-Nielsen (1998), Kou & Wang (2004), Carr *et al.* (2002), Seneta (2004), Behr & Pötter (2009), Ramezani & Zeng (2007), and others). On the other hand, in spite of their current importance, very few papers have considered intraday data. One of our main motivations in this work is to analyze whether pure Lévy models can still work well to fit the statistical properties of log returns at the intraday level.

As essentially any other model, a Lévy model will have limitations when

[1] That is, a model with infinitely many jumps during any finite time interval $[0, T]$.

working with very high-frequency transaction data and, hence, the question is rather to determine the scales where a Lévy model is a good probabilistic approximation of the underlying (extremely complex and stochastic) trading process. We propose to assess the suitability of the Lévy model by analyzing the signature plots of the point estimates at different sampling frequencies. It is plausible that an apparent stability of the point estimates for certain ranges of sampling frequencies provides evidence of the adequacy of the Lévy model at those scales. An earlier work along these lines is Eberlein & Özkan (2003), where this stability was empirically investigated using hyperbolic Lévy models and MLE (based on hourly data). Concretely, one of the main points therein was to estimate the model's parameters from daily mid-day log returns[2] and, then, measure the distance between the empirical density based on hourly returns and the one-hour density implied by the estimated parameters. It is found that this distance is approximately minimal among any other implied densities. In other words, if $f_\delta(\cdot;\theta_d^*)$ denotes the implied density of X_δ when using the parameters θ_d^* estimated from daily mid-day returns and if $f_{\mathrm{h}}^*(\cdot)$ denotes the empirical density based on hourly returns, then the distance between $f_\delta(\cdot;\theta_d^*)$ and f_{h}^* is minimal when δ is approximately one hour. Such a property was termed the "time consistency of Lévy processes".

In this paper, we further investigate the consistency of exponential Lévy models for a wide rage of intraday frequencies using intraday data of the U.S. equity market. Though natural differences due to sampling variation are to be expected, our empirical results under both models exhibit some very interesting common features across the different stocks we analyzed. We find that the estimator of the volatility parameter σ is quite stable for sampling frequencies as short as 20 minutes or less. For higher frequencies, the volatility estimates exhibit an abrupt tendency to increase (see Figure 6 below), presumably due to microstructure effects. In contrast, the kurtosis estimator is more sensitive to microstructure effects and a certain degree of stability is achieved only for mid range frequencies of 1 hour and more (see Figure 6 below). For higher frequencies, the kurtosis decreases abruptly. In fact, opposite to the smooth signature plot of σ at those scales, the kurtosis estimates consistently change by more than half when going from hourly to 30-minute log returns. Again, this phenomenon is presumably due to microstructure effects since the effect of an unaccounted continuous component will be expected to diminish when the sampling frequency increases.

One of the main motivations of Lévy models is that log returns follow ideal conditions for statistical inference in that case; namely, under a Lévy model the log returns at any frequency are independent with a common distribution. Due to this fact, it is arguable that it might be preferable to use a parsimonious model for which efficient estimation is feasible, rather than a very accurate model for which estimation errors will be intrinsically large. This is similar to the so-

[2]These returns are derived from prices recorded in the middle of the trading session. The idea behind the choice of these prices is to avoid the typically high volatility at the opening and closing of the trading session.

called model selection problem of statistics where a model with a high number of parameters typically enjoys a small mis-specification error but suffers from a high estimation variance due to the large number of parameters to estimate.

An intrinsic assumption in the previous paragraph is that standard estimation methods are indeed efficient in this high frequency data setting. This is, however, an overstatement (typically overlooked in the literature) since the population distribution of high-frequency sample data coming from a true Lévy model depends on the sampling frequency itself and, in spite of having more data, high frequency data does not necessarily imply better estimation results. Hence, another motivation for this work is to analyze the performance of the two most common estimators, namely the Method of Moments Estimators (MME) and the Maximum Likelihood Estimators (MLE), when dealing with high frequency data. As an additional contribution of this analysis, we also propose a simple novel numerical scheme for computing the MME. On the other hand, given the inaccessibility of closed forms for the MLE, we apply an unconstrained optimization scheme (Powell's method) to find them numerically.

By Monte Carlo simulations, we discover the surprising fact that neither high-frequency sampling nor MLE reduces the estimation error of the volatility parameter in a significant way. In other words, estimating the volatility parameter based on, say, daily observations has similar performance to doing the same based on, say, five minute observations. On the other hand, the estimation error of the parameter controlling the kurtosis of the model can be significantly reduced by using MLE or intraday data. Another conclusion is that the VG MLE is numerically unstable when working with ultra high frequency data while both the VG MME and the NIG MLE work quite well for almost any frequency.

The reminder of this article is organized as follows. In Section 2, we review the properties of the NIG and VG models. Section 3 introduces a simple and novel method to compute the moment estimators for the VG and the NIG distributions and also briefly describes the estimation method of maximum likelihood. Section 4 presents the finite-sample performance of the moment estimators and the maximum likelihood estimator via simulations. In Section 5, we present our empirical results using high-frequency transaction data from the U.S. equity market. The data was obtained from the NYSE TAQ database of 2005 trades via Wharton's WRDS system. For the sake of clarity and space, we only present the results for Intel and defer a full analysis of other stocks for a future publication. We finish with a section of conclusions and further recommendations.

2. The statistical models

2.1. Generalities of exponential Lévy models

Before introducing the specific models we consider in this paper, let us briefly motivate the application of Lévy processes in financial modeling. We refer the reader to the monographs of Cont & Tankov (2004) and Sato (1999) or the recent review papers Figueroa-López (2011) and Tankov (2011) for further information. Exponential (or Geometric) Lévy models are arguably the most natural generalization of the geometric Brownian motion intrinsic in the Black-Scholes option pricing model. A geometric Brownian motion (also called Black-Scholes model) postulates the following conditions about the price process $(S_t)_{t\geq 0}$ of a risky asset:

(i) The (log) return on the asset over a time period $[t, t+h]$ of length h, i.e.

$$R_{t,t+h} := \log \frac{S_{t+h}}{S_t},$$

is Gaussian with mean μh and variance $\sigma^2 h$ (independent of t);

(ii) Log returns on disjoint time periods are mutually independent;

(iii) The price path $t \to S_t$ is continuous; i.e. $\mathbb{P}(S_u \to S_t, \text{ as } u \to t, \text{ for all } t) = 1$.

The previous assumptions can equivalently be stated in terms of the so-called log return process $(X_t)_t$, denoted henceforth as

$$X_t := \log \frac{S_t}{S_0}.$$

Indeed, assumption (i) is equivalent to ask that the increment $X_{t+h} - X_t$ of the process X over $[t, t+h]$ is Gaussian with mean μh and variance $\sigma^2 h$. Assumption (ii) simply means that the increments of X over disjoint periods of time are independent. Finally, the last condition is tantamount to asking that X has continuous paths. Note that we can represent a general geometric Brownian motion in the form

$$S_t = S_0 e^{\sigma W_t + \mu t},$$

where $(W_t)_{t\geq 0}$ is the Wiener process. In the context of the above Black-Scholes model, a Wiener process can be defined as the log return process of a price process satisfying the Black-Scholes conditions (i)-(iii) with $\mu = 0$ and $\sigma^2 = 1$.

As it turns out, assumptions (i)-(iii) above are all controversial and believed not to hold true especially at the intraday level (see, e.g., Cont (2001) for a concise description of the most important features of financial data). The empirical distributions of log returns exhibit much heavier tails and higher kurtosis than a Gaussian distribution does and this phenomenon is accentuated when the frequency of returns increases. Independence is also questionable since, e.g., absolute log returns typically exhibit slowly decaying serial correlation. In other words, high-volatility events tend to cluster across time. Of course, continuity is just a convenient limiting abstraction to describe the high trading activity of liquid assets. In spite of its shortcomings, geometric Brownian motion could arguably be a suitable model to describe low frequency returns but not high frequency returns.

An exponential Lévy model attempts to relax the assumptions of the Black-Scholes model in a parsimonious manner. Indeed, a natural first step is to relax the Gaussian character of log returns by replacing it with an unspecified distribution as follows:

(i') The (log) return on the asset over a time period of length h has distribution F_h, depending only on the time span h.

This innocuous (still desirable) change turns out to be inconsistent with condition (iii) above in the sense that (ii)-(iii) together with (i') imply (i). Hence, we ought to relax (iii) as well if we want to keep (i'). The following is a natural compromise:

(iii') The paths $t \to S_t$ exhibit only discontinuities of first kind (jump discontinuities).

Summarizing, a Lévy model for the price process $(S_t)_{t\geq 0}$ of a risky asset satisfies conditions (i'), (ii), and (iii'). In the following section, we concentrate on two important and popular types of Lévy models.

2.2. *Variance Gamma and Normal Inverse Gaussian models*

The Variance Gamma (VG) and Normal Inverse Gaussian (NIG) Lévy models were proposed in Carr *et al.* (1998) and Barndorff-Nielsen (1998), respectively, to describe the log return process $X_t := \log S_t/S_0$ of a financial asset. Both models can be seen as a Wiener process with drift that is time-deformed by an independent random clock. That is, (X_t) has the representation

$$X_t = \sigma W(\tau(t)) + \theta\tau(t) + bt, \tag{2.1}$$

where $\sigma > 0, \theta, b \in \mathbb{R}$ are given constants, W is Wiener process, and τ is a suitable independent subordinator (non-decreasing Lévy process) such that

$$\mathbb{E}\tau(t) = t, \quad \text{and} \quad \mathrm{Var}(\tau(t)) = \kappa t.$$

In the VG model, $\tau(t)$ is Gamma distributed with scale parameter $\beta := \kappa$ and shape parameter $\alpha := t/\kappa$, while in the NIG model $\tau(t)$ follows an Inverse Gaussian distribution with mean $\mu = 1$ and shape parameter $\lambda = 1/(t\kappa)$. In the formulation (2.1), τ plays the role of a random clock aimed at incorporating variations in business activity through time.

The parameters of the model have the following interpretation (see (3.1) and (3.12) below):

1. σ dictates the overall variability of the log returns of the asset; In the symmetric case ($\theta = 0$), σ^2 is the variance of the log returns per unit time;

2. κ controls the kurtosis or tail heaviness of the log returns; In the symmetric case ($\theta = 0$), κ is the percentage excess kurtosis of log returns relative to the normal distribution multiplied by the time span;
3. b is a drift component in calendar time;
4. θ is a drift component in business time and controls the skewness of log returns;

The VG can be written as the difference of two Gamma Lévy processes,

$$X_t = X_t^+ - X_t^- + bt, \tag{2.2}$$

where X^+ and X^- are independent Gamma Lévy processes with respective parameters

$$\alpha^+ = \alpha^- = \frac{1}{\kappa}, \quad \beta\pm := \frac{\sqrt{\theta^2\kappa^2 + 2\sigma^2\kappa} \pm \theta\kappa}{2}.$$

One can see X^+ (resp. X^-) in (2.2) as the upward (resp. downward) movements in the asset's log return.

Under both models, the marginal density of X_t (which translates into the density of a log return over a time span t) is known in closed form. In the VG model, the probability density of X_t is given by

$$p_t(x) = \frac{\sqrt{2} e^{\frac{\theta(x-bt)}{\sigma^2}}}{\sigma\sqrt{\pi}\kappa^{\frac{t}{\kappa}}\Gamma(\frac{t}{\kappa})} \left(\frac{|x-bt|}{\sqrt{\frac{2\sigma^2}{\kappa} + \theta^2}} \right)^{\frac{t}{\kappa}-\frac{1}{2}} K_{\frac{t}{\kappa}-\frac{1}{2}} \left(\frac{|x-bt|\sqrt{\frac{2\sigma^2}{\kappa}+\theta^2}}{\sigma^2} \right), \tag{2.3}$$

where K is the modified Bessel function of the second kind (c.f. Carr *et al.* (1998)). The NIG model has marginal densities of the form

$$p_t(x) = \frac{t e^{\frac{t}{\kappa} + \frac{\theta(x-bt)}{\sigma^2}}}{\pi} \left(\frac{(x-bt)^2 + \frac{t^2\sigma^2}{\kappa}}{\frac{\theta^2}{\kappa\sigma^2} + \frac{1}{\kappa^2}} \right)^{-\frac{1}{2}} K_1 \left(\frac{\sqrt{(x-bt)^2 + \frac{t^2\sigma^2}{\kappa}}\sqrt{\frac{\sigma^2}{\kappa}+\theta^2}}{\sigma^2} \right), \tag{2.4}$$

Throughout the paper, we assume that the log return process $\{X_t\}_{t\geq 0}$ is sampled during a *fixed time interval* $[0, T]$ at evenly spaced times $t_i = i\delta_n$, $i = 1, \ldots, n$, where $\delta_n = T/n$. This sampling scheme is sometimes called *calendar time sampling* (c.f. Oomen (2006)). Under the assumption of independence and stationarity of the increments of X (conditions (i') and (ii) in Section 2.1), we have at our disposal a random sample

$$\Delta_i^n := \Delta_i^n X := X_{i\delta_n} - X_{(i-1)\delta_n}, \quad i = 1, \ldots, n, \tag{2.5}$$

of size n of the distribution $f_{\delta_n}(\cdot) := f_{\delta_n}(\cdot; \sigma, \theta, \kappa, b)$ of X_{δ_n}. Note that, in this context, a larger sample size n does not necessarily entail a greater amount of useful information about the parameters of the model. This is, in fact, one of the key questions in this paper: does the statistical performance of standard parametric methods improve under high-frequency observations? We will address this issue by simulation experiments in Section 4. For now, we introduce the statistical methods used in this article.

3. Parametric estimation methods

In this part, we review the most used parametric estimation methods: the method of moments and maximum likelihood. We also present a new computational method to find the moment estimators of the considered models. It is worth pointing out that both methods are known to be consistent under mild conditions if the number of observations at a *fixed* frequency (say, daily or hourly) are independent.

3.1. Method of moment estimators

In principle, the method of moments is a simple estimation method that can be applied to a wide range of parametric models. Also, the method of moment estimators (MME) are commonly used as initial points of numerical schemes used to find Maximum Likelihood Estimators (MLE), which are typically considered to be more efficient. Another appealing property of moment estimators is that they are known to be robust against possible dependence between log returns since their consistency is only a consequence of stationarity and ergodicity conditions of the log returns. In this section, we introduce a new method to compute the MME for the VG and NIG models.

Let us start with the VG model. The mean and first three central moments of a VG model are given in closed form as follows (see, e.g., Cont & Tankov, 2004, pp. 32 & 117):

$$
\begin{aligned}
&\mu_1(X_\delta) := \mathbb{E}(X_\delta) = (\theta + b)\delta, \\
&\mu_2(X_\delta) := \mathrm{Var}(X_\delta) = (\sigma^2 + \theta^2\kappa)\delta, \\
&\mu_3(X_\delta) := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2\theta\kappa + 2\theta^3\kappa^2)\delta, \\
&\mu_4(X_\delta) := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^4 = (3\sigma^4\kappa + 12\sigma^2\theta^2\kappa^2 + 6\theta^4\kappa^3)\delta + 3\mu_2(X_\delta)^2.
\end{aligned}
\tag{3.1}
$$

The MME is obtained by solving the system of equations resulting from substituting the central moments of X_{δ_n} in (3.1) by their corresponding sample estimators:

$$
\hat{\mu}_{k,n} := \frac{1}{n}\sum_{i=1}^{n}\left(\Delta_i^n - \bar{\Delta}^{(n)}\right)^k, \quad k \geq 2, \tag{3.2}
$$

where Δ_i^n is given as in (2.5) and $\bar{\Delta}^{(n)} := \sum_{i=1}^n \Delta_i^n/n$. However, solving the system of equations that defines the MME is not straightforward and, in general, one will need to rely on a numerical solution of the system. We now describe a novel simple method for this purpose. The idea is to write the central moments in terms of the quantity $\mathcal{E} := \theta^2\kappa/\sigma^2$. Concretely, we have the equations

$$
\begin{aligned}
&\mu_2(X_\delta) = \delta\sigma^2(1+\mathcal{E}), \quad \mu_3(X_\delta) = \delta\sigma^2\theta\kappa(3+2\mathcal{E}), \\
&\frac{\mu_4(X_\delta)}{3\mu_2^2(X_\delta)} - 1 = \frac{\kappa}{\delta}\,\frac{1+4\mathcal{E}+2\mathcal{E}^2}{(1+\mathcal{E})^2}.
\end{aligned}
$$

From these equations, it follows that

$$\frac{3\mu_3^2(X_\delta)}{\mu_2(X_\delta)\left(\mu_4(X_\delta)-3\mu_2^2(X_\delta)\right)}=\frac{\mathcal{E}\left(3+2\mathcal{E}\right)^2}{\left(1+4\mathcal{E}+2\mathcal{E}^2\right)\left(1+\mathcal{E}\right)}:=f(\mathcal{E}). \tag{3.3}$$

In spite of appearances, the above function $f(\mathcal{E})$ is a strictly increasing concave function from $(-1+2^{-1/2},\infty)$ onto $(-\infty,2)$ and, hence, the solution of the corresponding sample equation can be found efficiently using numerical methods. It remains to estimate the left hand side of (3.3). To this end, note that the left-hand side term can be written as $3\mathrm{Skw}(X_\delta)^2/\mathrm{Krt}(X_\delta)$, where Skw and Krt represent the population skewness and kurtosis:

$$\mathrm{Skw}(X_\delta):=\frac{\mu_3(X_\delta)}{\mu_2(X_\delta)^{3/2}},\quad \mathrm{Krt}(X_\delta):=\frac{\mu_4(X_\delta)}{\mu_2(X_\delta)^2}-3. \tag{3.4}$$

Finally, we just have to replace the population parameters by their empirical estimators:

$$\widehat{\mathrm{Var}}_n:=\frac{1}{n-1}\sum_{i=1}^n\left(\Delta_i^n-\bar{\Delta}^n\right)^2,\quad \widehat{\mathrm{Skw}}_n:=\frac{\hat{\mu}_{3,n}}{\hat{\mu}_{2,n}^{3/2}},\quad \widehat{\mathrm{Krt}}_n:=\frac{\hat{\mu}_{4,n}}{\hat{\mu}_{2,n}^2}-3. \tag{3.5}$$

Summarizing, the MME can be computed via the following numerical scheme:

1. Find (numerically) the solution $\hat{\mathcal{E}}_n^*$ of the equation

$$f(\mathcal{E})=\frac{3\,\widehat{\mathrm{Skw}}_n^{\,2}}{\widehat{\mathrm{Krt}}_n}; \tag{3.6}$$

2. Determine the MME using the following formulas:

$$\hat{\sigma}_n^2:=\frac{\widehat{\mathrm{Var}}_n}{\delta_n}\left(\frac{1}{1+\hat{\mathcal{E}}_n^*}\right),\ \hat{\kappa}_n:=\frac{\delta_n}{3}\,\widehat{\mathrm{Krt}}_n\left(\frac{(1+\hat{\mathcal{E}}_n^*)^2}{1+4\hat{\mathcal{E}}_n^*+2\hat{\mathcal{E}}_n^{*2}}\right), \tag{3.7}$$

$$\hat{\theta}_n:=\frac{\hat{\mu}_{3,n}}{\delta_n\hat{\sigma}_n^2\hat{\kappa}_n}\left(\frac{1}{3+2\hat{\mathcal{E}}_n^*}\right),\quad \hat{b}_n:=\frac{1}{\delta_n}\bar{\Delta}^n-\hat{\theta}_n=\frac{X_T}{T}-\hat{\theta}_n. \tag{3.8}$$

We note that the above estimators will exist if and only if the equation (3.6) admits a solution $\hat{\mathcal{E}}^*\in(-1+2^{-1/2},\infty)$, which is the case if and only if

$$\frac{3\,\widehat{\mathrm{Skw}}_n^{\,2}}{\widehat{\mathrm{Krt}}_n}<2.$$

Furthermore, the MME estimator $\widehat{\kappa}_n$ will be positive only if the sample kurtosis $\widehat{\mathrm{Krt}}_n$ is positive. It turns out that in simulations this condition is sometimes violated for small time horizons T and coarse sampling frequencies (say, daily or longer). For instance, using the parameter values (i) of Section 4.1 below and taking $T=125$ days (half a year) and $\delta_n=1$ day, about 80 simulations out of

1000 gave invalid $\hat{\kappa}$, while only 2 simulations result in invalid $\hat{\kappa}$ when $\delta_n = 1/2$ day.

Seneta (2004) proposes a simple approximation method built on the assumption that θ is typically small. In our context, Seneta's method is obtained by making the simplifying approximation $\hat{\mathcal{E}}_n^* \approx 0$ in the equations (3.7-3.8), resulting in the following estimators:

$$\hat{\sigma}_n^2 := \frac{\widehat{\mathrm{Var}}_n}{\delta_n}, \quad \hat{\kappa}_n := \frac{\delta_n}{3}\,\widehat{\mathrm{Krt}}_n, \tag{3.9}$$

$$\hat{\theta}_n := \frac{\hat{\mu}_{3,n}}{3\delta_n \hat{\sigma}_n^2 \hat{\kappa}_n} = \frac{\widehat{\mathrm{Skw}}_n(\widehat{\mathrm{Var}}_n)^{1/2}}{\delta_n \widehat{\mathrm{Krt}}_n}, \quad \hat{b}_n := \frac{X_T}{T} - \hat{\theta}_n. \tag{3.10}$$

Note that the estimators (3.9) are, in fact, the actual MME in the restricted symmetric model $\theta = 0$ and will indeed produce a good approximation of the MME estimators (3.7-3.8) whenever

$$\mathcal{Q}_n^* := \frac{3\,\widehat{\mathrm{Skw}}_n^{\,2}}{\widehat{\mathrm{Krt}}_n},$$

and, hence, $\hat{\mathcal{E}}_n^*$ is "very" small. This fact has been corroborated empirically by multiple studies using daily data as shown in Seneta (2004).

The formulas (3.9-3.10) have appealing interpretations as noted already by Carr *et al.* (1998). Namely, the parameter κ determines the percentage excess kurtosis in the log return distribution (i.e. a measure of the tail fatness compared to the normal distribution), σ dictates the overall volatility of the process, and θ determines the skewness. Interestingly, the estimator $\hat{\sigma}_n^2$ in (3.9) can be written as

$$\hat{\sigma}_n^2 = \frac{1}{T-\delta_n}\sum_{i=1}^{n}\left(X_{i\delta_n} - X_{(i-1)\delta_n} - \frac{X_T}{n}\right)^2 = \frac{1}{T-\delta_n}\overline{RV}_n + O\left(\frac{1}{n}\right),$$

where $\overline{RV}_n$ is the well-known realized variance defined by

$$\overline{RV}_n := \sum_{i=1}^{n}\left(X_{i\delta_n} - X_{(i-1)\delta_n}\right)^2. \tag{3.11}$$

Let us finish this section by considering the NIG model. In this setting, the mean and first three central moments are given by (see, e.g., Cont & Tankov, 2004, pp. 117):

$$\begin{aligned}
\mu_1(X_\delta) &:= \mathbb{E}(X_\delta) = (\theta + b)\delta, \\
\mu_2(X_\delta) &:= \mathrm{Var}(X_\delta) = (\sigma^2 + \theta^2\kappa)\delta, \\
\mu_3(X_\delta) &:= \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2\theta\kappa + 3\theta^3\kappa^2)\delta, \\
\mu_4(X_\delta) &:= \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^4 = (3\sigma^4\kappa + 18\sigma^2\theta^2\kappa^2 + 15\theta^4\kappa^3)\delta + 3\mu_2(X_\delta)^2.
\end{aligned} \tag{3.12}$$

Hence, the equation (3.3) takes the simpler form:

$$\frac{3\mu_3^2(X_\delta)}{\mu_2(X_\delta)\left(\mu_4(X_\delta)-3\mu_2^2(X_\delta)\right)}=\frac{9\mathcal{E}}{5\mathcal{E}+1}:=f(\mathcal{E}), \tag{3.13}$$

and the analogous equation (3.6) can be solved in closed form as

$$\hat{\mathcal{E}}_n^*=\frac{\widehat{\mathrm{Skw}}_n^{\,2}}{3\,\widehat{\mathrm{Krt}}_n-5\,\widehat{\mathrm{Skw}}_n^{\,2}}. \tag{3.14}$$

Then, the MME will be given by the following formulas:

$$\hat{\sigma}_n^2:=\frac{\widehat{\mathrm{Var}}_n}{\delta_n}\left(\frac{1}{1+\hat{\mathcal{E}}_n^*}\right),\ \hat{\kappa}_n:=\frac{\delta_n}{3}\,\widehat{\mathrm{Krt}}_n\left(\frac{1+\hat{\mathcal{E}}_n^*}{1+5\hat{\mathcal{E}}_n^*}\right), \tag{3.15}$$

$$\hat{\theta}_n:=\frac{\hat{\mu}_{3,n}}{\delta_n\hat{\sigma}_n^2\hat{\kappa}_n}\left(\frac{1}{3+3\hat{\mathcal{E}}_n^*}\right),\quad \hat{b}_n:=\frac{1}{\delta_n}\bar{\Delta}^n-\hat{\theta}_n=\frac{X_T}{T}-\hat{\theta}_n. \tag{3.16}$$

3.2. Maximum likelihood estimation

Maximum likelihood is one of the most widely used estimation methods, partly due to its theoretical efficiency when dealing with large samples. Given a random sample $\mathbf{x}=(x_1,\ldots,x_n)$ from a population distribution with density $f(\cdot|\theta)$ depending on a parameter $\theta=(\theta_1,\ldots,\theta_p)$, the method proposes to estimate θ with the value $\hat{\theta}=\hat{\theta}(\mathbf{x})$ that maximizes the so-called likelihood function

$$\mathcal{L}(\theta|\mathbf{x}):=\prod_{i=1}^{n}f(x_i|\theta).$$

When it exists, such a point estimate $\hat{\theta}(\mathbf{x})$ is called the Maximum Likelihood Estimator (MLE) of θ.

In principle, under a Lévy model, the increments of the log return process X (which corresponds to the log returns of the price process S) are independent with common distribution, say $f_\delta(\cdot|\theta)$, where δ represents the time span of the increments. As was pointed out earlier, independence is questionable for very high frequency log returns, but given that, for a large sample, likelihood estimation is expected to be robust against small dependences between returns, we can still apply likelihood estimation. The question is again to determine the scales where both the Lévy model is a good approximation of the underlying process and the maximum likelihood estimators are meaningful. As indicated in the introduction, it is plausible that the MLE's stability for certain range of sampling frequencies provides evidence of the adequacy of the Lévy model at those scales.

Another important issue is that, in general, the probability density f_δ is not known in a closed form or might be intractable. There are several approaches to deal with this issue such as numerically inverting the Fourier transform of f_δ via Fast Fourier Methods (see Carr *et al.* (2002)) or approximating f_δ using small-time expansions (see, e.g., Figueroa-López & Houdré (2009)). In the present article, we don't explore these approaches since the probability densities of the VG and NIG models are known in closed forms. However, given the inaccessibility of closed expressions for the MLE, we apply an unconstrained optimization scheme to find them numerically (see below for more details).

4. Finite-sample performance via simulations

4.1. Parameter values

We consider two sets of parameter values:

(i) $\sigma = \sqrt{6.447 * 10^{-5}} = .0080$; $\kappa = 0.422$; $\theta = -1.5 * 10^{-4}$; $b = 2.5750 * 10^{-4}$;
(ii) $\sigma = .0127$; $\kappa = 0.2873$; $\theta = 1.3 * 10^{-3}$; $b = -1.7 * 10^{-3}$;

The first set of parameters (i) is motivated by the empirical study reported in Seneta (2004) (see pp. 182) using the approximated MME introduced in Section 3.1 and daily returns of the Standard and Poor's 500 Index from 1977 to 1981. The second set of parameters (ii) is motivated by our own empirical results below using MLE and daily returns of INTC during 2005. Throughout, the time unit is a day and, hence, e.g., the estimated average rate of return per day of SP500 is

$$\mathbb{E}X(1) = \mathbb{E}\log\left(\frac{S_1}{S_0}\right) = \theta + b = 1.0750 * 10^{-4} \approx .1\%,$$

or $0.00010750 * 365 = 3.9\%$ per year.

4.2. Results

Below, we illustrate the finite-sample performance of the MME and MLE for both the VG and NIG models. The MME is computed using the algorithms described in Section 3.1. The MLE was computed using an unconstrained Powell's method[3] started at the exact MME. We use the closed form expressions for the density functions (2.3-2.4) in order to evaluate the likelihood function.

4.2.1. Variance Gamma

We compute the sample mean and sample standard deviation of the VG MME and the VG MLE for different sampling frequencies. Concretely, the time span δ

[3]We employ a MATLAB implementation due to Giovani Tonel obtained through MATLAB Central (http://www.mathworks.com/matlabcentral/fileexchange/).

between consecutive observations is taken to be $1/36, 1/18, 1/12, 1/6, 1/3, 1/2, 1$ (in days), which will correspond to $10, 20, 30$ minutes, $1, 2, 3$ hours and 1 day (assuming a trading period of six hours per day). Figure 1 plots the sampling mean $\bar{\hat{\sigma}}_\delta$ and the bands $\bar{\hat{\sigma}}_\delta \pm \text{std}(\hat{\sigma}_\delta)$ against the different time spans δ as well as the corresponding graphs for κ, based on 100 simulations of the VG process on $[0, 3 * 252]$ (namely, 3 years) with the parameter values (i) above. Similarly, Figure 2 shows the results corresponding to the parameter values (ii) with a time horizon of $T = 252$ days and time spans δ equal to 10, 20, and 30 minutes, and also, 1/6, 1/4, 1/3, 1/2, and 1 days, assuming this time a trading period of six and a half hours per day and taking 200 simulations. These are our conclusions:

1. The MME for σ performs as well as the computationally more expensive MLE for all the relevant frequencies. Even though increasing the sampling frequency slightly reduces the standard error, the net gain is actually very small even for very high frequencies and, hence, does not justify the use of high-frequency data to estimate σ.
2. The estimation for κ is quite different: Using either high-frequency data or maximum likelihood estimation results in significant reductions of the standard error (by more than 4 times when using both).
3. The computation of the MLE presents numerical issues (easy to detect) for very high sampling frequencies (say $\delta < 1/6$).
4. Disregarding the numerical issues and extrapolating the pattern of the graphs when $\delta \to 0$, we can conjecture that the MLE $\hat{\sigma}$ is not consistent when $\delta \to 0$ for a fixed time horizon T, while the MLE $\hat{\kappa}$ appears to be a consistent estimator for κ. Both of these points will be investigated in a future publication.

For completeness, we also illustrates in Figure 3 the performance of the estimators for b and θ for the parameter values (ii) based again on 200 simulations during $[0, 252]$ with time spans of 10, 20, and 30 minutes, and 1/6, 1/4, 1/3, 1/2, and 1 days. There seems to be some gain in efficiency when using MLE and higher sampling frequencies in both cases but the respective standard errors level off for δ small, suggesting that neither estimator is consistent for fixed time horizon. One surprising feature is that the MLE estimators in both cases do not seem to exhibit any numerical issues for very small δ in spite of being based on the same simulations as those used to obtain $\hat{\sigma}$ and $\hat{\kappa}$.

4.2.2. Normal Inverse Gaussian

We now show the estimation results for the NIG model. Here, we take sampling frequencies of 5, 10, 20, and 30 seconds, also 1, 5, 10, 20, and 30 minutes, as well as 1, 2, and 3 hours, and finally 1 day (assuming a trading period of 6 hours). Figure 4 plots the sampling mean $\bar{\hat{\sigma}}_\delta$ and bands $\bar{\hat{\sigma}}_\delta \pm \text{std}(\hat{\sigma}_\delta)$ against the different time spans δ andthe corresponding graphs for κ, based on 100 simulations of the NIG process on $[0, 3 * 252]$ with the parameter values (i)

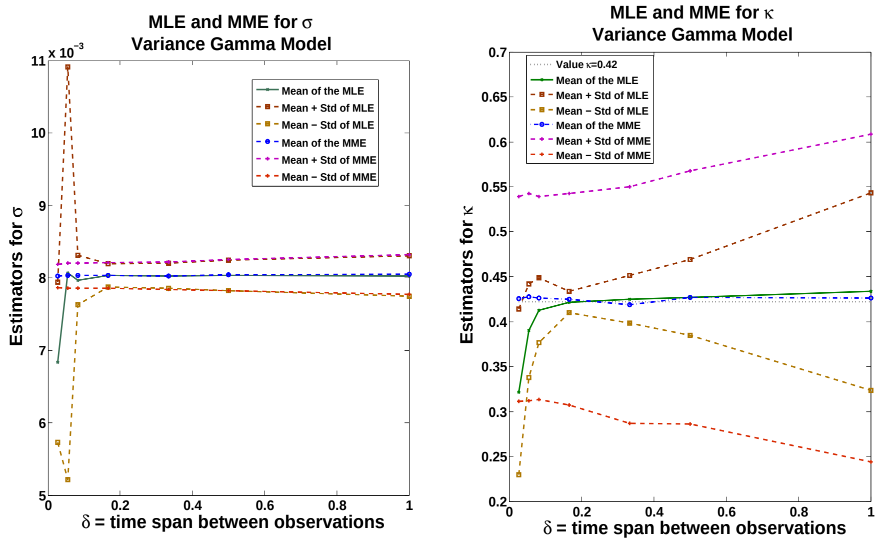

FIG 1. *Sampling mean and standard error of the MME and MLE for the parameters σ and κ based on 100 simulations of the VG model with values $T = 252{*}3$, $\sigma = \sqrt{6.447 * 10^{-5}} = .0080$; $\kappa = 0.422$; $\theta = -1.5 * 10^{-4}$; $b = 2.5750 * 10^{-4}$.*

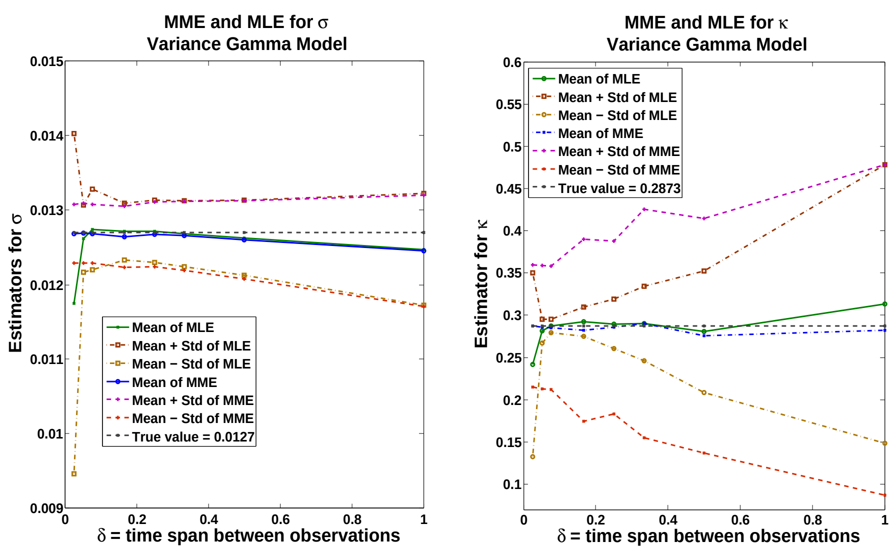

FIG 2. *Sampling mean and standard error of the MME and MLE for the parameters σ and κ based on 200 simulations with values $T = 252$, $\sigma = .0127$; $\kappa = 0.2873$; $\theta = 1.3 * 10^{-3}$; $b = -1.7 * 10^{-3}$.*

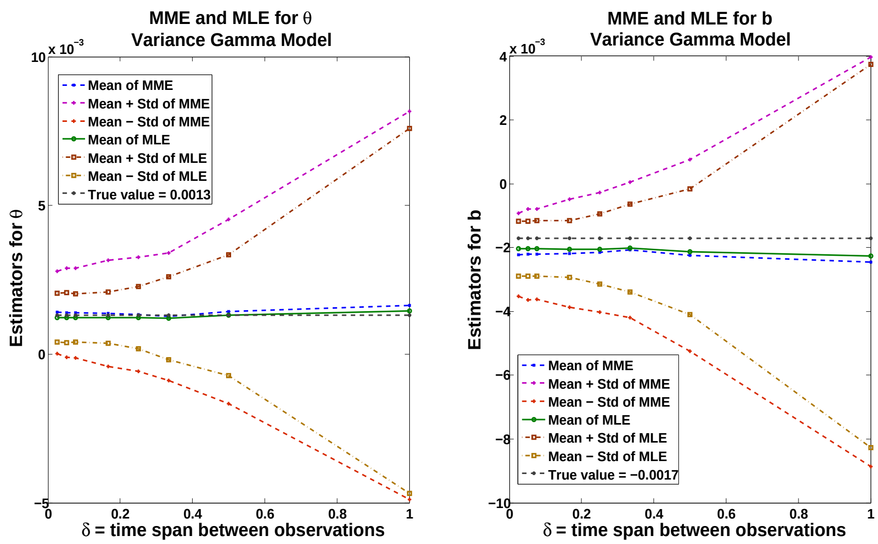

FIG 3. *Sampling mean and standard error of the MME and MLE for the parameters θ and b based on 200 simulations with values $T = 252$, $\sigma = .0127$; $\kappa = 0.2873$; $\theta = 1.3 * 10^{-3}$; $b = -1.7 * 10^{-3}$.*

above. The results are similar to those of the VG model. In the case of σ, neither MLE nor high-frequency data seem to do better than standard moment estimators and daily data. For κ, the estimation error can be reduced as much as 4 times when using high-frequency data and maximum likelihood estimation. The most striking conclusion is that the MLE for the NIG model does not show any numerical issues when dealing with very high-frequency. Indeed, we are able to obtain results for even 5 second time spans (although the computational time increases significantly in this case).

5. Empirical results

5.1. The data and data pre-processing

The data was obtained from the NYSE TAQ database of 2005 trades via Wharton's WRDS system. For the sake of clarity and space, we focus on the analysis of only one stock, even though other stocks were also analyzed for this study. We pick Intel (INTC) stock due to its high liquidity (based on the number of trades or ticks). The raw data was preprocessed as follows. Records of trades were kept if the TAQ field CORR indicated that the trade was "regular" (namely, it was not corrected, changed, signaled as cancelled, or signaled as an error). In addition, the condition field was use as a filter. Trades were kept if they were regular way trades, that is, trades that had no stated conditions (COND="" or

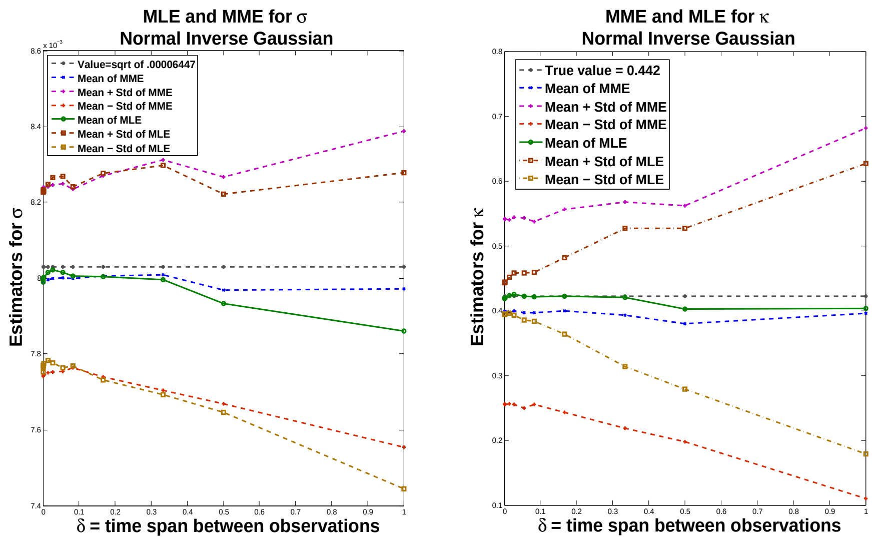

FIG 4. *Sampling mean and standard error of the MME and MLE for the parameters σ and κ based on 100 simulations of the NIG model with values $T = 252 * 3$, $\sigma = \sqrt{6.447 * 10^{-5}} = .0080$; $\kappa = 0.422$; $\theta = -1.5 * 10^{-4}$; $b = 2.5750 * 10^{-4}$.*

COND="*"). A secondary filter was subsequently applied to eliminate some of the remaining incorrect trades. First, for each trading day, the empirical distribution of the absolute value of the first difference of prices was determined. Next, the 99.9th percentile of these daily absolute differences was obtained. Finally, a trade was eliminated if, in magnitude, the difference of the price from the prior price was at least twice the 99.9th percentile of that day's absolute differences and this difference was reversed on the following trade. Figure 5 illustrates the Intel stock prices before (left panel) and after processing (right panel).

5.2. MME and MLE results

The exact and approximated MMEs described in Section 3.1 were applied to the log returns of the stocks at different frequencies ranging from 10 seconds up to 1 day. Subsequently, we apply the unconstrained Powell's optimization method to find the MLE estimator. In each case, the starting point for the optimization routine was set equal to the exact MME. Tables 1 to 4 show the estimation results under both models together with the log likelihood values using a time horizon of one year. Figure 6 show the graphs of the NIG MLE and approximated NIG MME against the sampling frequency δ based on observations during $T = 1$ year, $T = 6$ months, and $T = 3$ months, respectively.

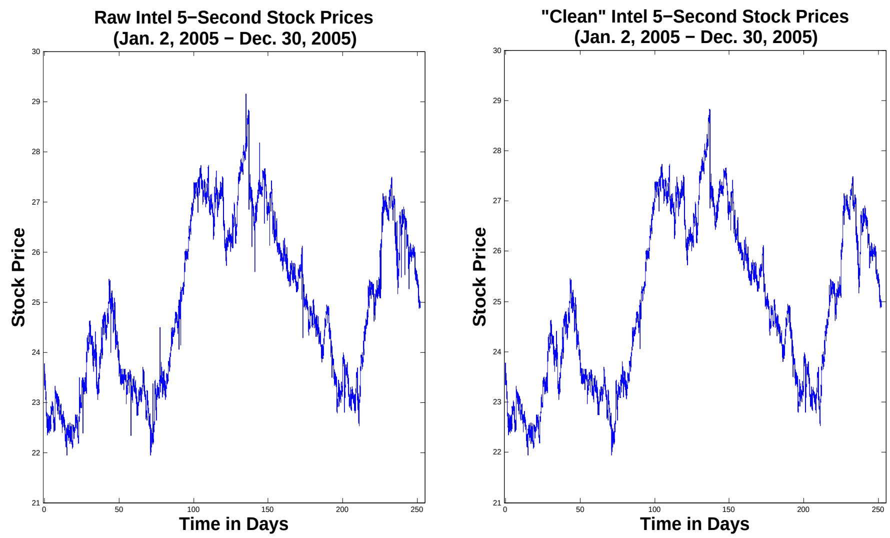

FIG 5. *Intel stock prices during 2005 before and after preprocessing.*

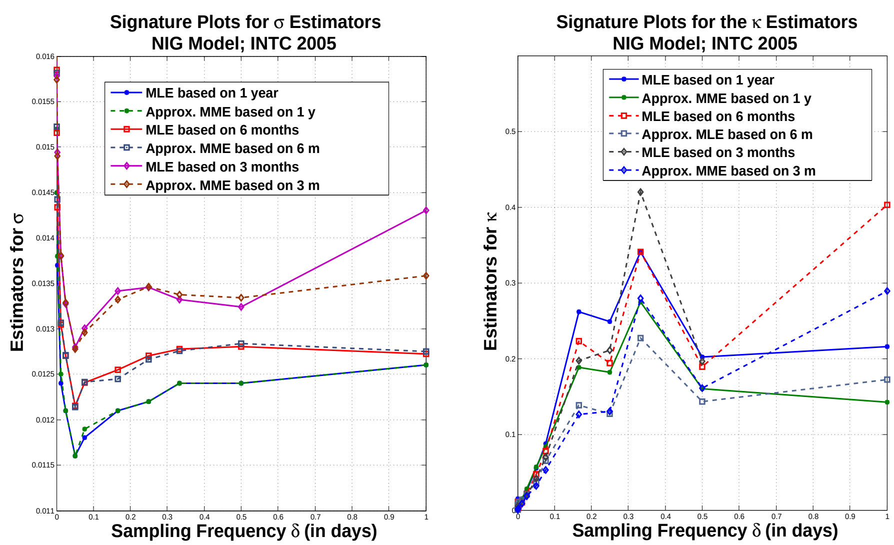

FIG 6. *Signature plots for the MLE and MME for σ under a NIG model based on different time horizons.*

5.3. Discussion of empirical results

In spite of certain natural differences due to sampling variation, the empirical results under both models exhibit some very interesting common features that we now summarize:

1. The estimation of σ is quite stable for "midrange" frequencies ($\delta \geq 20$ minutes), exhibiting a slight tendency to decrease when δ decreases from 1 day to 10 minutes, before showing a pronounce and clear tendency to increase for small time spans ($\delta = 10$ minutes and less). This increasing tendency is presumably due to the influence of microstructure effects.
2. The point estimators for κ are less stable than those for σ but still their values are relatively "consistent" for mid range frequencies of 1 hour and more. This consistency of $\hat{\kappa}$ abruptly changes when δ moves from 1/6 of a day to 30 minutes, at which point a reduction of about half is experienced under both models. To illustrate how unlikely such a behavior is in our models, we consider the simulation experiment of Figure 2 and find out that in only 1 out of the 200 simulations the exact MME estimator for κ increased by more than twice its value when δ goes from 30 minutes to 1/6 of a day (only 3 out 200 simulations showed an increment of more than 1.5). In none of the 200 simulation, the MLE estimator for κ increased more than 1.5 its value when δ goes from 30 minutes to 1/6 of a day. For the NIG model, using the simulations of Figure 4, we found out that in only 3 out 100 simulations the MME estimator for κ increased by more than 1.2 when δ goes from 30 minutes to 1/6 of a day (it never increased for more than 1.5). Such a jump in the empirical results could be interpreted as a consequence of microstructure effects;
3. According to our previous simulation analysis, the estimators for κ are more reliable when δ gets smaller. Hence, we recommend using the value of the estimator for δ as small as possible, but still in the range where we suspect that microstructure effects are relatively low. For instance, one can propose to take $\hat{\kappa} = 0.1662$ under the VG model (resp. $\hat{\kappa} = 0.2621$ under the NIG model), or alternatively, one could average the MLE estimators for $\delta > 1/2$.
4. Under both models, the estimators for κ show a certain tendency to decrease as δ gets very small (less than 30 minutes).
5. Given the higher sensitivity of κ to microstructure effects, one could use the values of this estimator to identify the range of frequencies where a Lévy model is adequate and microstructure effects are still low. In the case of INTC, one can recommend using a Lévy model to describe log returns higher than one hour. As an illustration of the goodness of fit, Figures 7 shows the empirical histograms of $\delta = 1/6$ returns against the fitted VG model and NIG model using Maximum Likelihood Estimation. We also show the fitted Gaussian distributions in each case. Both models show very good fit. The graphs in log scale, useful to check the fit at the tails, are shown in Figure 8.

6. Conclusion

Certain parametric classes of exponential Lévy models have appealing features for modeling intraday financial data. In this article, we lean towards choosing a parsimonious model with few parameters that has natural financial interpretation, rather than a complex over-parameterized model. Even though, in principle, a complex model will provide a better fit of the observed empirical features of financial data, the intrinsically less accurate estimation or calibration of such a model might render it less useful in practice. By contrast, we consider here two simple and well-known models for the analysis of intraday data: the Variance Gamma model of Carr *et al.* (1998) and the Normal Inverse Gaussian model of Barndorff-Nielsen (1998). These models require one additional parameter, when compared to the two-parameter Black-Scholes model, that controls the tail thickness of the log return distribution.

As essentially any other model, a Lévy model will have limitations when working with very high-frequency transaction data and, hence, in our opinion the real problem is to determine the sampling frequencies at which a specific Lévy model will be a "good" probabilistic approximation of the underlying trading process. In this paper we put forward an intuitive statistical method to solve this problem. Concretely, we propose to assess the suitability of the Lévy model by analyzing the signature plots of statistical point estimates at different sampling frequencies. It is plausible that an apparent stability of the point estimates for certain ranges of sampling frequencies will provide evidence of the adequacy of the Lévy model at those scales. At least based on our preliminary empirical analysis, we find that a Lévy model seems a reasonable model for log returns as frequent as hourly and that the kurtosis estimate is a more sensitive indicator of microstructure effects in the data than the volatility estimate, which exhibits a very stable behavior for sampling time spans as small as 20 minutes.

We also studied the in-fill numerical performance of the two most widely used parametric estimators: the method of moment estimators and the maximum likelihood estimation. We discover that neither high frequency sampling nor maximum likelihood estimation significantly reduces the estimation error of the volatility parameter of the model. Hence, we can "safely" estimate the volatility parameter using a simple moment estimator applied to daily closing prices. The estimation of the kurtosis parameter is quite different. In that case, using either high-frequency data or maximum likelihood estimation can result in significant reductions of the standard error (by more than 4 times when using both). Both of these results appear to be new in the statistical literature of high frequency data.

The problem of finding the MLE based on very high frequency data remains a challenging numerical problem, even if closed form expressions are available as it is the case of the NIG and VG models. On the contrary, in this paper, we propose a simple numerical method to find the MME of the NIG and VG models. Moment estimators are particularly appealing in the context of high-frequency data since their consistency does not require independence between log returns but only stationarity and ergodicity conditions.

δ	20 min	30 min	1/6	1/4	1/3	1/2	1
$\hat{\kappa}$	0.0354	0.0542	0.1662	0.1724	0.2342	0.2098	0.2873
$\hat{\sigma}$	0.0115	0.0117	0.0120	0.0121	0.0123	0.0125	0.0127
$\hat{\theta}$	0.0010	0.0023	0.0019	0.0011	0.0020	0.0020	0.0013
$\hat{b}$	-0.0014	-0.0027	-0.0023	-0.0015	-0.0024	-0.0023	-0.0017
$\log \mathcal{L}$	2.2485e+4	1.4266e+4	6.0015e+3	3.7580e+3	2.6971e+3	1.6783e+3	745.8689
$\hat{\kappa}$	0.0571	0.0834	0.1839	0.1804	0.2694	0.1579	0.1383
$\hat{\sigma}$	0.0116	0.0119	0.0120	0.0121	0.0123	0.0124	0.0125
$\hat{\theta}$	0.0016	0.0010	0.0032	0.0019	0.0024	0.0028	0.0041
$\hat{b}$	-0.0020	-0.0014	-0.0036	-0.0022	-0.0028	-0.0032	-0.0045
$\log \mathcal{L}$	2.2438e+4	1.4243e+4	5.9946e+3	3.7578e+3	2.6966e+3	1.6780e+3	745.5981
$\hat{\kappa}$	0.0573	0.0835	0.1887	0.1819	0.2749	0.1603	0.1423
$\hat{\sigma}$	0.0116	0.0119	0.0121	0.0122	0.0124	0.0124	0.0126
$\hat{\theta}$	0.0016	0.0010	0.0031	0.0018	0.0024	0.0027	0.0040
$\hat{b}$	-0.0020	-0.0014	-0.0035	-0.0022	-0.0027	-0.0031	-0.0043
$\log \mathcal{L}$	2.2437e+4	1.4243e+4	5.9942e+3	3.7577e+3	2.6965e+3	1.6781e+3	745.6023

TABLE 1
INTC: VG MLE (Top), Exact VG MME (Middle), and Approx. VG MME (Bottom).

δ	10 sec	20 sec	30 sec	1 min	5 min	10 min
$\hat{\kappa}$	0.0128	0.0112	0.0183	0.0354	0.0501	0.0191
$\hat{\sigma}$	0.0465	0.0300	0.0303	0.0293	0.0173	0.0120
$\hat{\theta}$	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0002
$\hat{b}$	0.0000	-0.0000	0.0000	0.0000	0.0000	-0.0002
$\log \mathcal{L}$	5.2980e+6	2.4338e+6	1.5115e+6	6.6256e+5	1.0540e+5	4.7949e+4
$\hat{\kappa}$	0.0010	0.0023	0.0052	0.0080	0.0153	0.0282
$\hat{\sigma}$	0.0169	0.0152	0.0145	0.0138	0.0125	0.0121
$\hat{\theta}$	-0.0001	0.0014	0.0025	-0.0040	-0.0013	0.0011
$\hat{b}$	-0.0003	-0.0018	-0.0029	0.0036	0.0009	-0.0015
$\log \mathcal{L}$	4.3254e+6	2.0063e+6	1.2823e+6	5.8998e+5	1.0203e+5	4.7897e+4
$\hat{\kappa}$	0.0010	0.0023	0.0052	0.0081	0.0153	0.0282
$\hat{\sigma}$	0.0169	0.0152	0.0145	0.0138	0.0125	0.0121
$\hat{\theta}$	-0.0001	0.0014	0.0025	-0.0040	-0.0013	0.0011
$\hat{b}$	-0.0003	-0.0018	-0.0029	0.0036	0.0009	-0.0015
$\log \mathcal{L}$	4.3254e+6	2.0063e+6	1.2823e+6	5.8987e+5	1.0203e+5	4.7897e+4

TABLE 2
INTC: VG MLE (Top), Exact VG MME (Middle), and Approx. VG MME (Bottom).

δ	20 min	30 min	1/6	1/4	1/3	1/2	1
$\hat{\kappa}$	0.0557	0.0874	0.2621	0.2494	0.3412	0.2024	0.2159
$\hat{\sigma}$	0.0116	0.0118	0.0121	0.0122	0.0124	0.0124	0.0126
$\hat{\theta}$	0.0019	0.0017	0.0017	0.0012	0.0018	0.0019	0.0019
$\hat{b}$	-0.0022	-0.0021	-0.0021	-0.0016	-0.0022	-0.0023	-0.0022
$\log \mathcal{L}$	2.2498e+4	1.4274e+4	5.9988e+3	3.7575e+3	2.6969e+3	1.6777e+3	745.6436
$\hat{\kappa}$	0.0570	0.0833	0.1791	0.1789	0.2640	0.1554	0.1343
$\hat{\sigma}$	0.0116	0.0119	0.0120	0.0121	0.0123	0.0124	0.0125
$\hat{\theta}$	0.0016	0.0010	0.0033	0.0019	0.0025	0.0028	0.0042
$\hat{b}$	-0.0020	-0.0014	-0.0037	-0.0022	-0.0028	-0.0032	-0.0046
$\log \mathcal{L}$	2.2498e+4	1.4274e+4	5.9952e+3	3.7564e+3	2.6963e+3	1.6775e+3	745.5409
$\hat{\kappa}$	0.0573	0.0835	0.1887	0.1819	0.2749	0.1603	0.1423
$\hat{\sigma}$	0.0116	0.0119	0.0121	0.0122	0.0124	0.0124	0.0126
$\hat{\theta}$	0.0016	0.0010	0.0031	0.0018	0.0024	0.0027	0.0040
$\hat{b}$	-0.0020	-0.0014	-0.0035	-0.0022	-0.0027	-0.0031	-0.0043
$\log \mathcal{L}$	2.2498e+4	1.4274e+4	5.9957e+3	3.7563e+3	2.6964e+3	1.6776e+3	745.5465

TABLE 3
INTC: NIG MLE (Top), Exact NIG MME (Middle), and Approx. NIG MME (Bottom).

δ	10 sec	20 sec	30 sec	1 min	5 min	10 min
$\hat{\kappa}$	0.1349	0.0061	0.0012	0.0024	0.0125	0.0220
$\hat{\sigma}$	0.0341	0.0190	0.0149	0.0134	0.0119	0.0114
$\hat{\theta}$	-0.0002	0.0007	0.0086	0.0095	0.0042	0.0037
$\hat{b}$	-0.0000	-0.0009	-0.0088	-0.0097	-0.0044	-0.0038
$\log \mathcal{L}$	3.8974e+6	1.8740e+6	1.2188e+6	5.8072e+5	1.0206e+5	4.7957e+4
$\hat{\kappa}$	0.0003	0.0007	0.0012	0.0031	0.0157	0.0252
$\hat{\sigma}$	0.0194	0.0161	0.0148	0.0134	0.0119	0.0114
$\hat{\theta}$	0.0194	0.0187	0.0160	0.0134	0.0070	0.0042
$\hat{b}$	-0.0196	-0.0189	-0.0162	-0.0136	-0.0072	-0.0044
$\log \mathcal{L}$	3.8863e+6	1.8718e+6	1.2135e+6	5.7856e+5	1.0204e+5	4.7955e+4
$\hat{\kappa}$	0.0003	0.0007	0.0012	0.0031	0.0160	0.0255
$\hat{\sigma}$	0.0194	0.0161	0.0148	0.0134	0.0120	0.0114
$\hat{\theta}$	0.0194	0.0187	0.0159	0.0132	0.0069	0.0042
$\hat{b}$	-0.0196	-0.0188	-0.0161	-0.0134	-0.0070	-0.0044
$\log \mathcal{L}$	3.8863e+6	1.8718e+6	1.2135e+6	5.7850e+5	1.0204e+5	4.7955e+4

TABLE 4
INTC: NIG MLE (Top), Exact NIG MME (Middle), and Approx. NIG MME (Bottom).

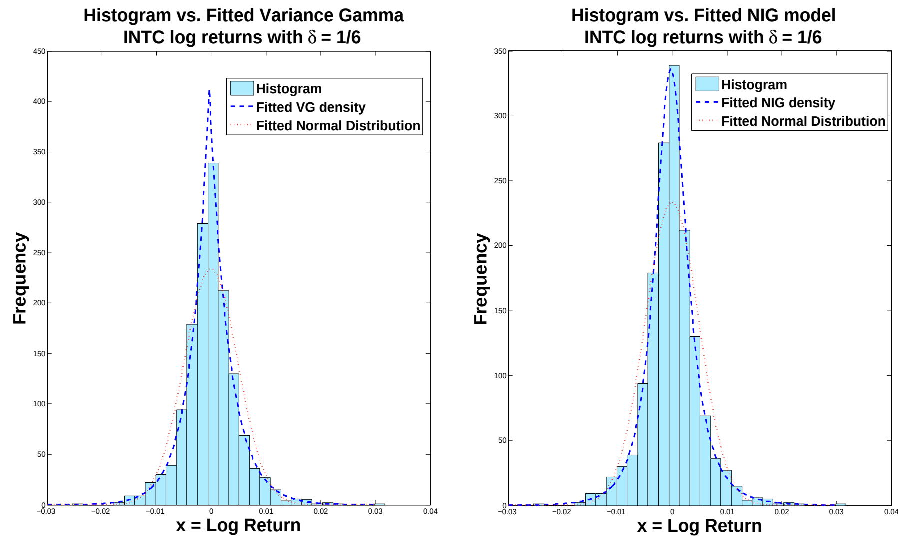

FIG 7. *Histograms of INTC returns for $\delta = 1/6$ and the fitted VG and NIG models using Maximum Likelihood Estimation.*

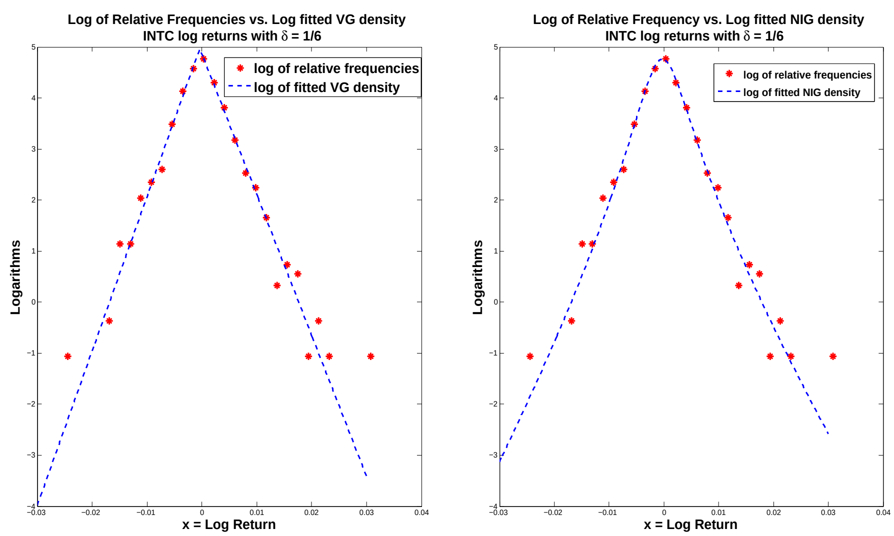

FIG 8. *Logarithm of the histograms of INTC returns for $\delta = 1/6$ and the fitted VG and NIG models using Maximum Likelihood Estimation.*

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