Jump-diffusion models driven by Lévy processes

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Abstract: During the past and this decade, a new generation of continuous-time financial models has been intensively investigated in a quest to incorporate the so-called *stylized empirical features of asset prices* like fat-tails, high kurtosis, volatility clustering, and leverage. Modeling driven by "memoryless homogeneous" jump processes (Lévy processes) constitutes one of the most viable directions in this enterprise. The basic principle is to replace the underlying Brownian motion of the Black-Scholes model with a type of jump-diffusion process. In this chapter, the basic results and tools behind jump-diffusion models driven by Lévy processes are covered, providing an accessible overview, coupled with their financial applications and relevance. The material is drawn upon recent monographs (cf. [18], [46]) and papers in the field.

Keywords and phrases: Lévy processes, Poisson random measures, jump-diffusion financial models, exponential Lévy models.

1. An overview of financial models with jumps

The seminal Black-Scholes model [10] provides a framework to price options based on the fundamental concepts of hedging and absence of arbitrage. One of the key assumptions of the Black-Scholes model is that the stock price process $t \to S_t$ is given by a *geometric Brownian motion* (GBM), originally proposed by Samuelson [45]. Concretely, the time-t price of the stock is postulated to be given by

$$S_t = S_0 e^{\sigma W_t + \mu t}, \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. This model is plausible since Brownian motion is the model of choice to describe the evolution of a random measurement whose value is the result of a large-number of small shocks occurring through time with high-frequency. This is indeed the situation with the log return process $X_t = \log(S_t/S_0)$ of a stock, whose value at

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a given time t (not "very" small) is the superposition of a high number of small movements driven by a large number of agents posting bid and ask prices almost at "all times".

The limitations of the GBM were well-recognized almost from its inception. For instance, it well known that the time series of log returns, say $\log\{S_\Delta/S_0\}, \ldots, \log\{S_{k\Delta}/S_{(k-1)\Delta}\}$, exhibit *leptokurtic distributions* (i.e. fat tails with high kurtosis distributions), which are inconsistent with the Gaussian distribution postulated by the GBM. As expected the discrepancy from the Gaussian distribution is more marked when Δ is small (say a day and smaller). Also, the volatility, as measured for instance by the square root of the realized variance of log returns, exhibits *clustering* and *leverage* effects, which contradict the random-walk property of a GBM. Specifically, when plotting the time series of log returns against time, there are periods of high variability followed by low variability periods suggesting that high volatility events "cluster" in time. Leverage refers to a tendency towards a volatility growth after a sharp drop in prices, suggesting that volatility is negatively correlated with returns. These and other *stylized statistical features* of asset returns are widely known in the financial community (see e.g. [17] and [9] for more information). In the risk-neutral world, it is also well known that the Black-Scholes implied volatilities of call and put options are not flat neither with respect to the strike nor to the maturity, as it should be under the Black-Scholes model. Rather implied volatilities exhibit smile or smirk curve shapes.

In a quest to incorporate the stylized properties of asset prices, many models have been proposed during the last and this decade, most of them derived from natural variations of the Black-Scholes model. The basic idea is to replace the Brownian motion W in (1), with another related process such as a Lévy process, a Wiener integral $\int_0^t \sigma_s dW_s$, or a combination of both, leading to a "*jump-diffusion model*" or a *semimartingale* model. The simplest jump-diffusion model is of the form

$$S_t := S_0 e^{\sigma W_t + \mu t + Z_t}, \tag{2}$$

where $Z := \{Z_t\}_{t \geq 0}$ is a "pure-jump" Lévy process. Equivalently, (2) can be written as

$$S_t := S_0 e^{X_t}, \tag{3}$$

where X_t is a general Lévy process. Even this simple extension of the GBM, called *geometric Lévy model* or *exponential Lévy model*, is able to incorporate several stylized features of asset prices such as heavy tails, high-kurtosis, and asymmetry of log returns. There are other reasons in support of incorporating *jumps* in the dynamics of the stock prices. On one hand, certain event-driven information often produces "sudden" and "sharp" price changes at discrete unpredictable times. Second, in fact stock prices are made up of discrete trades occurring through time at a very high frequency. Hence, processes exhibiting infinitely many jumps in any finite time horizon $[0, T]$ are arguably better approximations to such high-activity stochastic processes.

Merton [35], following Press [40], proposed one of the earliest models of the form (2), taking a compound Poisson process Z with normally distributed jumps (see Section 2.1). However, earlier Mandelbrot [34] had already proposed a pure-jump model driven by a stable Lévy process Z. Merton's model is considered to exhibit light tails as all exponential moments of the densities of $\log(S_t/S_0)$ are finite, while Mandelbrot's model exhibit very heavy tails with not even finite second moments. It was during the last decade that models exhibiting appropriate tail behavior were proposed. Among the better known models are the *variance Gamma model* of [13], the *CGMY model* of [11], and the *generalized hyperbolic motion* of [6, 7] and [19, 20]. We refer to Kyprianou et. al. [32, Chapter 1] and Cont and Tankov [18, Chapter 4] for more extensive reviews and references of the different types of geometric Lévy models in finance.

The geometric Lévy model (2) cannot incorporate volatility clustering and leverage effects due to the fact that log returns will be independent identically distributed. To cope with this shortcoming, two general classes of models driven by Lévy processes have been proposed. The first approach, due to Barndorff-Nielsen and Shephard (see e.g. [7] and references therein), proposes a stochastic volatility model of the form

$$S_t := S_0 e^{\int_0^t b_u du + \int_0^t \sigma_u dW_u}, \tag{4}$$

where σ is a stationary non-Gaussian Ornstein-Uhlenbeck process

$$\sigma_t^2 = \sigma_0^2 + \int_0^t \alpha \sigma_s^2 ds + Z_{\alpha t},$$

driven by a subordinator Z (i.e. a non-decreasing Lévy process) (see Shephard [47] and Andersen and Benzoni [3] for two recent surveys on these and other related models). The second approach, proposed by Carr et. al. [12, 14], introduces stochastic volatility via a random clock as follows:

$$S_t = S_0 e^{Z_{\tau(t)}}, \quad \text{with} \quad \tau(t) := \int_0^t r(u) du. \tag{5}$$

The process τ plays the role of a "business" clock which could reflect non-synchronous trading effects or a "cumulative measure of economic activity". Roughly speaking, the rate process r controls the volatility of the process; for instance, in time periods where r is "high", the "business time" τ runs faster resulting in more frequent jump times. Hence, positive *mean-revering diffusion processes* $\{r(t)\}_{t\geq 0}$ are plausible choices to incorporate volatility clustering.

To account for the leverage phenomenon, different combinations of the previous models have been considered leading to semimartingale models driven by Wiener and Poisson random measures. A very general model in this direction assumes that the log return process

$X_t := \log(S_t/S_0)$ is given as follows (c.f. [27], [48]):

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|\leq 1} \delta(s,x)\bar{M}(ds,dx) + \int_0^t \int_{|x|>1} \delta(s,x) M(ds,dx)$$
$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_{|x|\leq 1} \tilde{\delta}(s,x)\bar{M}(ds,dx) + \int_0^t \int_{|x|>1} \tilde{\delta}(s,x) M(ds,dx),$$

where W is a $d-$dimensional Wiener process, M is the *jump measure* of an independent Lévy process Z, defined by

$$M(B) := \#\{(t, \Delta Z_t) \in B : t > 0 \text{ such that } \Delta Z_t \neq 0\},$$

and $\bar{M}(dt,dx) := M(dt,dx) - \nu(dx)dt$ is the compensate Poisson random measure of Z, where ν is the Lévy measure of Z. The integrands (b, σ, etc.) are random processes themselves, which could even depend on X and σ leading to a system of stochastic differential equations. One of the most active research fields in this very general setting is that of statistical inference methods based on high-frequency (intraday) financial data. Some of the researched problems include the prediction of the integrated volatility process $\int_0^t \sigma_s^2 ds$ or of the Poisson integrals $\int_0^t \int_{\mathbb{R}\setminus\{0\}} g(x) M(dx,ds)$ based on realized variations of the process (see e.g. [27, 28], [33], [50, 51], [37], [8]), testing for jumps ([8], [37], [1]), and the estimation in the presence of "microstructure" noise ([2], [38, 39]).

In this work, the basic methods and tools behind jump-diffusion models driven by Lévy processes are covered. The chapter will provide an accessible overview of the probabilistic concepts and results related to Lévy processes, coupled whenever is possible with their financial application and relevance. Some of the topics include: construction and characterization of Lévy processes and Poisson random measures, statistical estimation based on high- and low-frequency observations, density transformation and risk-neutral change of measures, arbitrage-free option pricing and integro-partial differential equations. The material is drawn upon recent monographs (c.f. [18], [46]) and recent papers in the field.

2. Distributional properties and statistical estimation of Lévy processes

2.1. Definition and fundamental examples

A Lévy process is a probabilistic model for an unpredictable measurement X_t that evolves in time t, in such a way that the change of the measurement in disjoint time intervals of equal duration, say $X_{s+\Delta} - X_s$ and $X_{t+\Delta} - X_t$ with $s + \Delta \leq t$, are independent from one another but with identical distribution. For instance, if S_t represents the time-t price of an asset and X_t is the *log return during* $[0, t]$, defined by

$$X_t = \log(S_t/S_0),$$

then the previous property will imply that daily or weekly log returns will be independent from one another with common distribution. Formally, a Lévy process is defined as follows:

Definition 1. *A **Lévy process** $X = \{X_t\}_{t\geq 1}$ is a $\mathbb{R}^d$-valued stochastic process (collection of random vectors in $\mathbb{R}^d$ indexed by time) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

(i) $X_0 = 0$;
(ii) *X has **independent increments**: $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for any $0 \leq t_0 < \cdots < t_n$;*
(iii) *X has **stationary increments**: the distribution of $X_{t+\Delta} - X_t$ is the same as X_Δ, for all $t, \Delta \geq 0$;*
(iv) *its paths are right-continuous with left-limits (rcll);*
(v) *it has **no fixed jump-times**; that is, $\mathbb{P}(\Delta X_t \neq 0) = 0$, for any time t.*

The last property can be replaced by asking that X is continuous in probability, namely, $X_s \xrightarrow{\mathbb{P}} X_t$, as $s \to t$, for any t. Also, if X satisfies all the other properties except (iv), then there exists a rcll version of the process (see e.g. Sato [46]).

There are three fundamental examples of Lévy processes that deserve some attention: Brownian motion, Poisson process, and compound Poisson process.

Definition 2. *A (standard) Brownian motion W is a real-valued process such that (i) $W_0 = 0$, (ii) it has independent increments, (iii) $W_t - W_s$ has normal distribution with mean 0 and variance $t - s$, for any $s < t$, and (iv) it has continuous paths.*

It turns out that the only real Lévy processes with continuous paths are of the form $X_t = \sigma W_t + bt$, for constants $\sigma > 0$ and b.

A Poisson process is another fundamental type of Lévy process that is often used as building blocks of other processes.

Definition 3. *A Poisson process N is an integer-valued process such that (i) $N_0 = 0$, (ii) it has independent increments, (iii) $N_t - N_s$ has Poisson distribution with parameter $\lambda(t - s)$, for any $s < t$, and (iv) its paths are rcll. The parameter λ is called the intensity of the process.*

The Poisson process is frequently used as a model to count events of certain type (say, car accidents) occurring randomly through time. Concretely, suppose that $T_1 < T_2 < \ldots$ represent random occurrence times of a certain event and let N_t be the number of events occurring by time t:

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t\}}. \tag{6}$$

Then, if the events occur independently from one another, homogeneously in time, and with an intensity of λ events per unit time, $\{N_t\}_{t\geq 0}$ given by (6) will be approximately a Poisson process with intensity λ. This fact is a consequence of the Binomial approximation to the Poisson distribution (see, e.g., Feller [21] for this heuristic construction of a Poisson process). It turns out that any Poisson process can be written in the form (6) with $\{T_i\}_{i\geq 1}$ (called arrival times) such that the waiting times

$$\tau_i := T_i - T_{i-1},$$

are independent exponential r.v.'s with common mean $1/\lambda$ (so, the bigger the λ, the smaller the expected waiting time between arrivals and the higher the intensity of arrivals).

To introduce the last fundamental example, the compound Poisson process, we recall the concept of probability distribution. Given a random vector J in $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathbb{P})$, the distribution of J is the mapping ρ defined on sets $A \subset \mathbb{R}^d$ as follows:

$$\rho(A) := \mathbb{P}(J \in A).$$

Thus, $\rho(A)$ measures the probability that the random vector J belongs to the set A. A *compound Poisson process* with jump distribution ρ and jump intensity λ is a process of the form

$$Z_t := \sum_{i=1}^{N_t} J_i,$$

where $\{J_i\}_{i\geq 1}$ are independent with common distribution ρ and N is a Poisson process with intensity λ that is independent of $\{J_i\}_i$. When $d = 1$, one can say that the compound Poisson process $Z := \{Z_t\}_{t\geq 0}$ is like a Poisson process with random jump sizes independent from one another. A compound Poisson process is the only Lévy process that has piece-wise constant paths with finitely-many jumps in any time interval $[0, T]$. Note that the distribution of the compound Poisson process Z is characterized by the finite measure:

$$\nu(A) := \lambda \rho(A), \quad A \subset \mathbb{R}^d,$$

called the *Lévy measure* of Z. Furthermore, for any finite measure ν, one can associate a compound Poisson process Z with Lévy measure ν (namely, the compound Poisson process with intensity of jumps $\lambda := \nu(\mathbb{R}^d)$ and jump distribution $\rho(dx) := \nu(dx)/\nu(\mathbb{R}^d)$).

For future reference, it is useful to note that the characteristic function of Z_t is given by

$$\mathbb{E}e^{i\langle u, Z_t\rangle} = \exp\left\{t\int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle}-1\right)\nu(dx)\right\} \tag{7}$$

Also, if $\mathbb{E}|J_i| = \int |x|\rho(dx) < \infty$, then $\mathbb{E}Z_t = t\int x\rho(dx)$ and the so-called *compensated compound Poisson process* $\bar{Z}_t := Z_t - \mathbb{E}Z_t$ has characteristic function

$$\mathbb{E}e^{i\left\langle u,\bar{Z}_t\right\rangle} = \exp\left\{t\int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle}-1-i\left\langle u,x\right\rangle\right)\nu(dx)\right\}. \tag{8}$$

One of the most fundamental results establishes that any Lévy process can be approximated arbitrarily close by the superposition of a Brownian motion with drift, $\sigma W_t + bt$, and an independent compound Poisson process Z. The reminder $R_t := X_t - (\sigma W_t + bt + Z_t)$ is a pure-jump Lévy process with jump sizes smaller than say an $\varepsilon > 0$, which can be taken arbitrarily small. The previous fundamental fact is a consequence of the Lévy-Itô decomposition that we review in Section 3.2.

2.2. Infinitely divisible distributions and the Lévy-Khintchine formula

The marginal distributions of a Lévy process X are *infinitely-divisible*. A random variable ξ is said to be *infinitely divisible* if for each $n \geq 2$, one can construct n i.i.d. r.v's . $\xi_{n,1}, \ldots, \xi_{n,n}$ such that

$$\xi \overset{\mathfrak{D}}{=} \xi_{n,1} + \cdots + \xi_{n,n}.$$

That X_t is infinitely divisible is clear since

$$X_t = \sum_{k=0}^{n-1}\left(X_{(k+1)t/n} - X_{kt/n}\right),$$

and $\{X_{(k+1)t/n} - X_{kt/n}\}_{k=0}^{n-1}$ are i.i.d. The class of infinitely divisible distributions is closely related to limits in distribution of an array of row-wise i.i.d. r.v.'s:

Theorem 1 (Kallenberg [30])**.** *ξ is infinitely divisible iff for each n there exists i.i.d. random variables $\{\xi_{n,k}\}_{k=1}^{k_n}$ such that*

$$\sum_{k=1}^{k_n}\xi_{n,k} \xrightarrow{\mathfrak{D}} \xi, \quad \text{as} \quad n\to\infty.$$

In term of the characteristic function $\varphi_\xi(u) := \mathbb{E}e^{i\langle u,\xi\rangle}$, ξ is infinitely divisible if and only if $\varphi_\xi(u) \neq 0$, for all u, and its distinguished n^{th}-root $\varphi_\xi(u)^{1/n}$ is the characteristic function of some other variable for each n (see Lemma 7.6 in [46]). This property of the characteristic function turns out to be sufficient to determine its form in terms of three "parameters" (A, b, ν), called the *Lévy triplet* of ξ, as defined below.

Theorem 2 (Lévy-Khintchine formula). *ξ is infinitely divisible iff*

$$\mathbb{E}e^{i\langle u,\xi\rangle} = \exp\left\{ i\langle b,u\rangle - \frac{1}{2}\langle u, Au\rangle + \int \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbf{1}_{|x|\leq 1}\right)\nu(dx)\right\}, \tag{9}$$

for some symmetric nonnegative-definite matrix A, a vector $b \in \mathbb{R}^d$, and a measure ν (called the Lévy measure) on $\mathbb{R}^d_0 := \mathbb{R}^d \backslash \{0\}$ such that

$$\int_{\mathbb{R}^d_0} (|x|^2 \wedge 1)\nu(dx) < \infty. \tag{10}$$

Moreover, all triplets (A, b, ν) with the stated properties may occur.

The following remarks are important:

Remark 1. *The previous result implies that the time-t marginal distribution of a Lévy process $\{X_t\}_{t\geq 0}$ is identified with a Lévy triplet (A_t, b_t, ν_t). Given that X has stationary and independent increments, it follows that $\mathbb{E}e^{i\langle u,X_t\rangle} = \left\{\mathbb{E}e^{i\langle u,X_1\rangle}\right\}^t$, for any rational t and by the right-continuity of X, for any real t. Thus, if (A, b, ν) is the Lévy triplet of X_1, then $(A_t, b_t, \nu_t) = t(A, b, \nu)$ and*

$$\varphi_{X_t}(u) := \mathbb{E}e^{i\langle u,X_t\rangle} = e^{t\psi(u)}, \qquad \textit{where} \tag{11}$$

$$\psi(u) := i\langle b,u\rangle - \frac{1}{2}\langle u, Au\rangle + \int \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbf{1}_{|x|\leq 1}\right)\nu(dx). \tag{12}$$

The triple (A, b, ν) is called the Lévy or characteristic triplet of the Lévy process X.

Remark 2. *The exponent (12) is called the* Lévy exponent *of the Lévy process $\{X_t\}_{t\geq 0}$. We can see that its first term is the Lévy exponent of the Lévy process bt. The second term is the Lévy exponent of the Lévy process ΣW_t, where $W = (W^1, \ldots, W^d)^T$ are $d-$independent Wiener processes and Σ is a $d \times d$ lower triangular matrix in the Cholesky decomposition $A = \Sigma\Sigma^T$. The last term in the Lévy exponent can be decomposed into two terms:*

$$\psi^{cp}(u) = \int_{|x|>1} \left(e^{i\langle u,x\rangle} - 1\right)\nu(dx), \quad \psi^{lccp}(u) = \int_{|x|\leq 1} \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\right)\nu(dx).$$

The first term above is the Lévy exponent of a compound Poisson process X^{cp} with Lévy measure $\nu_1(dx) := \mathbf{1}_{|x|>1}\nu(dx)$ (see (7)). The exponent ψ^{lccp} corresponds to the limit in distribution of compensated compound Poisson processes. *Concretely, suppose that $X^{(\varepsilon)}$ is a compound Poisson process with Lévy measure $\nu_\varepsilon(dx) := \mathbf{1}_{\varepsilon<|x|\leq 1}\nu(dx)$, then the process $X_t^{(\varepsilon)} - \mathbb{E}X_t^{(\varepsilon)}$ converges in distribution to a process with characteristic function* $\exp\{t\psi^{lccp}\}$ *(see (8)). Lévy-Khintchine formula implies that, in distribution, X is the superposition of four independent Lévy processes as follows:*

$$X_t \stackrel{\mathfrak{D}}{=} \underbrace{bt}_{\textit{Drift}} + \underbrace{\Sigma W_t}_{\textit{Brownian part}} + \underbrace{X_t^{cp}}_{\textit{Cmpnd. Poisson}} + \underbrace{\lim_{\varepsilon\searrow 0}\left(X_t^{(\varepsilon)} - \mathbb{E}X_t^{(\varepsilon)}\right)}_{\textit{Limit of cmpstd cmpnd Poisson}}, \tag{13}$$

where equality is in the sense of finite-dimensional distributions. The condition (10) on ν guarantees that the X^{cp} is indeed well defined and the compensated compound Poisson converges in distribution.

In the rest of this section, we go over some other fundamental distributional properties of the Lévy process and their applications.

2.3. Short-term distributional behavior

The characteristic function (11) of X determines uniquely the Lévy triple (A, b, ν). For instance, the uniqueness of the matrix A is a consequence of the following result:

$$\lim_{h\to 0} h \cdot \log \varphi_{X_t}\left(h^{-1/2}u\right) = -\frac{t}{2}\langle u, Au\rangle; \tag{14}$$

see pp. 40 in [46]. In term of the process X, (14) implies that

$$\left\{\frac{1}{\sqrt{h}}X_{ht}\right\}_{t\geq 0} \stackrel{\mathfrak{D}}{\longrightarrow} \{\Sigma W_t\}_{t\geq 0}, \quad \text{as } h \to 0. \tag{15}$$

where $W = (W^1, \ldots, W^d)^T$ are $d-$independent Wiener processes and Σ is a lower triangular matrix such that $A = \Sigma\Sigma^T$.

From a statistical point of view, (15) means that, when $\Sigma \neq 0$, the short-term increments $\{X_{(k+1)h} - X_{kh}\}_{k=1}^n$, properly scaled, behave like the increments of a Wiener process. In the context of the exponential Lévy model (34), the result (15) will imply that the log returns of the stock, properly scaled, are normally distributed when the Brownian component of the Lévy process X is non-zero. This property is not consistent with the empirical heavy tails of high-frequency financial returns. Recently, Rosiński [44] proposes a pure-jump class of Lévy processes, called *tempered stable (TS) Lévy processes*, such that

$$\left\{\frac{1}{h^{1/\alpha}}X_{ht}\right\}_{t\geq 0} \xrightarrow{\mathfrak{D}} \{Z_t\}_{t\geq 0}, \quad \text{as } h \to 0, \tag{16}$$

where Z is a stable process with index $\alpha < 2$.

2.4. Moments and short-term moment asymptotics

Let $g : \mathbb{R}^d \to \mathbb{R}_+$ be a nonnegative locally bounded function and X be a Lévy process with Lévy triplet (A, b, ν). The expected value $\mathbb{E}g(\xi)$ is called the g-moment of a random variable ξ. Let us now consider submultiplicative or subadditive moment functions g. Recall that a nonnegative locally bounded function g is submultiplicative (resp. subadditive) if there exists a constant $K > 0$ such that $g(x + y) \leq Kg(x)g(y)$ (resp. $g(x + y) \leq K(g(x) + g(y))$), for all x, y. Examples of this kind of functions are $g(x_1, \dots, x_d) = |x_j|^p$, for $p \geq 1$, and $g(x_1, \dots, x_d) = \exp\{|x_j|^\beta\}$, for $\beta \in (0, 1]$. In the case of a compound Poisson process, it is easy to check that

$\mathbb{E}g(X_t) < \infty$, for any $t > 0$ if and only if $\int_{|x|>1} g(x)\nu(dx) < \infty$.

The previous fact holds for general Lévy processes (see Kruglov [31] and Sato [46, Theorem 25.3]). In particular, $X(t) := (X_1(t), \dots, X_d(t)) := X_t$ has finite mean if and only if $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$. In that case, by differentiation of the characteristic function, it follows that

$$\mathbb{E}X_j(t) = t\left(\int_{\{|x|>1\}} x_j\nu(dx) + b_j\right),$$

Similarly, $\mathbb{E}|X(t)|^2 < \infty$ if and only if $\int_{\{|x|>1\}} |x|^2\nu(dx) < \infty$, in which case,

$$\operatorname{Cov}(X_j(t), X_k(t)) = t\left(A_{jk} + \int x_jx_k\nu(dx)\right).$$

The two above equations show the connection between the the Lévy triplet (A, b, ν), and the mean and covariance of the process. Note that the variance rate $\operatorname{Var}(X_j(t))/t$ remains constant over time. It can also be shown that the kurtosis is inversely proportional to time t. In the risk-neutral world, these properties are not empirically supported under the exponential Lévy model (2), which rather support a model where both measurements increase with time t (see e.g. [12] and references therein).

The Lévy measure ν controls the short-term ergodic behavior of X. Namely, for any bounded continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ vanishing on a neighborhood of the origin, it holds that

$$\lim_{t\to 0} \frac{1}{t}\,\mathbb{E}\varphi(X_t) = \int \varphi(x)\nu(dx); \tag{17}$$

cf. Sato [46, Corollary 8.9]. For a real Lévy processes X with Lévy triplet (σ^2, b, ν), (17) can be extended to incorporate unbounded functions and different behaviors at the origin. Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is $\nu-$continuous such that $|\varphi| \leq g$ for a subadditive or submultiplicative function $g : \mathbb{R} \to \mathbb{R}_+$. Furthermore, fixing $I := \{r \geq 0 : \int (|x|^r \wedge 1)\, \nu(dx) < \infty\}$, assume that φ exhibits any of the following behaviors as $x \to 0$:

(a) i. $\varphi(x) = o(|x|^2)$;
  ii. $\varphi(x) = O(|x|^r)$, for some $r \in I \cap (1, 2)$ and $\sigma = 0$;
  iii. $\varphi(x) = o(|x|)$, $1 \in I$ and $\sigma = 0$;
  iv. $\varphi(x) = O(|x|^r)$, for some $r \in I \cap (0, 1)$, $\sigma = 0$, and $\bar{b} := b - \int_{|x|\leq 1} x\nu(dx) = 0$.

(b) $\varphi(x) \sim x^2$;

(c) $\varphi(x) \sim |x|$ and $\sigma = 0$.

Building on results in [50] and [28], [23] proves that

$$\lim_{t\to 0} \frac{1}{t}\, \mathbb{E}\varphi(X_t) := \begin{cases} \int \varphi(x)\nu(dx), & \text{if (a) holds,} \\ \sigma^2 + \int \varphi(x)\nu(dx), & \text{if (b) holds,} \\ |\bar{b}| + \int \varphi(x)\nu(dx), & \text{if (c) holds.} \end{cases} \tag{18}$$

Woerner [50] and also Figueroa-López [22] used the previous short-term ergodic property to show the consistency of the statistics

$$\hat{\beta}^{\pi}(\varphi) := \frac{1}{t_n} \sum_{k=1}^{n} \varphi\left(X_{t_k} - X_{t_{k-1}}\right), \tag{19}$$

towards the integral parameter $\beta(\varphi) := \int \varphi(x)\nu(dx)$, when $t_n \to \infty$ and $\max\{t_k - t_{k-1}\} \to 0$, for test functions φ as in (a). When $\nu(dx) = s(x)dx$, Figueroa-López [22] applied the estimators (19) to analyze the asymptotic properties of nonparametric *sieve-type* estimators $\hat{s}$ for s. The problem of model selection was analyzed further in [24, 25], where it was proved that sieve estimators $\widetilde{s}_T$ can match the rate of convergence of the minimax risk of estimators $\hat{s}$. Concretely, it turns out that

$$\limsup_{T\to\infty} \frac{\mathbb{E}\|s - \widetilde{s}_T\|^2}{\inf_{\hat{s}} \sup_{s\in\Theta} \mathbb{E}\|s - \hat{s}\|^2} < \infty,$$

where $[0, T]$ is the time horizon over which we observe the process X, Θ is certain class of smooth functions, and the infimum in the denominator is over all estimators $\hat{s}$ which are based on whole trajectory $\{X_t\}_{t\leq T}$. The optimal rate of the estimator $\widetilde{s}_T$ is attained by choosing appropriately the dimension of the sieve and the sampling frequency in function of T and the smoothness of the class of functions Θ.

2.5. Extraction of the Lévy measure

The Lévy measure ν can be inferred from the characteristic function $\varphi_{X_t}(u)$ of the Lévy process (see, e.g, Sato [46, pp. 40-41]). Concretely, by first recovering $\langle u, Au\rangle$ from (14), one can obtain

$$\Psi(u) := \log \varphi_{X_1}(u) + \frac{1}{2}\langle u, Au\rangle .$$

Then, it turns out that

$$\int_{[-1,1]^d} (\Psi(u) - \Psi(u+w))\, dw = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\tilde{\nu}(dx), \tag{20}$$

where $\tilde{\nu}$ is the finite measure

$$\tilde{\nu}(dx) := 2^d \left(1 - \prod_{j=1}^{d} \frac{\sin x_j}{x_j}\right)\nu(dx).$$

Hence, ν can be recovered from the inverse Fourier transform of the left-hand side of (20).

The above method can be applied to devise non-parametric estimation of the Lévy measure by replacing the Fourier transform φ_{X_1} by its empirical version:

$$\hat{\varphi}_{X_1}(u) := \frac{1}{n}\sum_{k=1}^{n} \exp\left\{i\langle u, X_k - X_{k-1}\rangle\right\}.$$

given discrete observations $X_1, \ldots, X_n$ of the process. Recently, similar nonparametric methods have been proposed in the literature to estimate the Lévy density $s(x) = \nu(dx)/dx$ of a real Lévy process X (c.f. [36], [16], and [26]). For instance, based on the increments $X_1 - X_0, \ldots, X_n - X_{(n-1)}$, Neumann and Reiss [36] consider a nonparametric estimator for s that minimizes the distance between the "population" characteristic function $\varphi_{X_1}(\cdot; s)$ and the empirical characteristic function $\hat{\varphi}_{X_1}(\cdot)$. By appropriately defining the distance metric, Neumann and Reiss (2008) showed the consistency of the proposed estimators. Another approach, followed for instance by Watteel and Kulperger [49] and Comte and Genon-Catalot [16], relies on an "explicit" formula for the Lévy density s in terms of the derivatives of the characteristic function φ_{X_1}. For instance, under certain regularity conditions,

$$\mathcal{F}\left(xs(x)\right)(\cdot) = -i\frac{\varphi'_{X_1}(\cdot)}{\varphi_{X_1}(\cdot)},$$

where $\mathcal{F}(f)(u) = \int e^{iux} f(x)dx$ denotes the Fourier transform of a function f. Hence, an estimator for s can be built by replacing ψ by a smooth version of the empirical estimate $\hat{\varphi}_{X_1}$ and applying inverse Fourier transform $\mathcal{F}^{-1}$.

3. Path decomposition of Lévy processes

In this part, we show that the construction in (13) holds true a.s. (not only in distribution) and draw some important consequences. The fundamental tool for this result is a probabilistic characterization of the random points $\{(t, \Delta X_t) : t \text{ s.t. } \Delta X_t \neq 0\}$ as a *Poisson point process* on the semi-plane $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$. Due to this fact, we first review the properties of Poisson random measures, which are also important building blocks of financial models.

3.1. Poisson random measures and point processes

Definition 4. *Let S be a Borel subset of $\mathbb{R}^d$, let $\mathcal{S}$ be the set of Borel subsets of S, and let m be a $\sigma-$finite measure on S. A collection $\{M(B) : B \in \mathcal{S}\}$ of $\bar{\mathbb{Z}}_+$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a* ***Poisson random measure*** *(PRM) (or process) on S with mean measure m if*

(1) for every $B \in \mathcal{S}$, $M(B)$ is a Poisson random variable with mean $m(B)$;
(2) if $B_1, \dots, B_n \in \mathcal{S}$ are disjoint, then $M(B_1), \dots, M(B_1)$ are independent;
(3) for every sample outcome $\omega \in \Omega$, $M(\cdot;\omega)$ is a measure on $\mathcal{S}$.

Above, we used some basic terminology of real analysis. For all practical purposes, Borel sets of $\mathbb{R}^d$ are those subsets that can be constructed from basic operations (complements, countable unions, and intersections) of elementary sets of the form $(a_1, b_1] \times \dots (a_d, b_d]$. A measure m is a mapping from $\mathcal{S}$ to $[0, \infty]$ such that

$$m(\emptyset) = 0, \quad \text{and} \quad m\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} m(B_i),$$

for any mutually disjoints Borel sets $\{B_i\}_{i\geq 1}$. A measure is said to be σ-finite if there exists mutually disjoints $\{B_i\}_{i\geq 1}$ such that $\mathbb{R}^d = \bigcup_{i=1}^{\infty} B_i$ and $m(B_i) < \infty$, for any i.

It can be proved that (a.s.), a Poisson random measure $M(\cdot;\omega)$ is an atomic measure; that is, there exist countably many (random) points $\{\mathbf{x}_i\}_i \subset S$ (called atoms) such that

$$M(B) = \#\{i : \mathbf{x}_i \in B\} = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(B). \tag{21}$$

Similarly, if a sequence of finitely many or countably many random points $\{\mathbf{x}_i\}_i$ is such that the measure (21) satisfies (1)-(3) above, then we say that $\{\mathbf{x}_i\}_i$ is a **Poisson point process** on S with mean measure m. The following is a common procedure to construct a realization of a Poisson random measure or point process :

1. Suppose that $B_1, B_2, \dots$ is a partition of S such that $m(B_j) < \infty$.
2. Generate $n_j \sim Poiss(m(B_j))$.
3. Independently, generate n_j-points, say $\{\mathbf{x}_i^j\}_{i=1}^{n_j}$, according to the distribution $m(\cdot)/m(B_j)$.
4. Define $M(B) = \#\{(i,j) : \mathbf{x}_i^j \in B\}$.

3.1.1. Transformation of Poisson random measures

Among the most useful properties of PRM is that certain transformations of a Poisson point process are still a Poisson point process. The following is the simplest version:

Proposition 1. *Suppose that $T : S \to S' \subset \mathbb{R}^{d'}$ is a one-to-one measurable function. Then, the random measure associated with the transformed points $\mathbf{x}_i' := T(\mathbf{x}_i)$, namely $M'(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i'}(\cdot)$, is also a Poisson random measure with mean measure $m'(B) := m(\{\mathbf{x} : T(\mathbf{x}) \in B\})$.*

The following result shows that a *marked Poisson point process* is still a Poisson point process. Suppose that we associate a $\mathbb{R}^{d'}$-valued score $\mathbf{x}_i'$ to each point $\mathbf{x}_i$ of M. The scores are assigned independently from one another. The distribution of the scores can actually depend on the point $\mathbf{x}_i$. Concretely, let $\sigma(\mathbf{x}, d\mathbf{x}')$ be a probability measure on $S' \subset \mathbb{R}^{d'}$, for each $\mathbf{x} \in S$ (hence, $\sigma(\mathbf{x}, S') = 1$). For each i, generate a r.v. $\mathbf{x}_i'$ according $\sigma(\mathbf{x}_i, d\mathbf{x}')$ (*independently from any other variable*). Consider the so-called *marked Poisson process*

$$M'(\cdot) = \sum_{i=1}^{\infty} \delta_{(\mathbf{x}_i, \mathbf{x}_i')}(\cdot).$$

Proposition 2. *M' is a Poisson random measure on $\mathcal{S} \times \mathcal{S}'$ with mean measure*

$$m'(d\mathbf{x}, d\mathbf{x}') = \sigma(\mathbf{x}, d\mathbf{x}')m(d\mathbf{x}).$$

As an example, consider the following experiment. We classify the points of the Poisson process M into k different types. The probability that the point $\mathbf{x}_i$ is of type j is $p_j(\mathbf{x}_i)$ (necessarily $p_j(\cdot) \in [0,1]$), independently from any other classification. Let $\{\mathbf{y}_i^j\}$ be the points of $\{\mathbf{x}_i\}$ of type j and let M^j be the counting measure associated with $\{\mathbf{y}_i^j\}$:

$$M^j := \sum \delta_{\{\mathbf{y}_i^j\}}$$

We say that the process M^1 is constructed from M by *thinning*.

Proposition 3. *$M^1, \dots, M^k$ are independent Poisson random measures with respective mean measures $m_1(d\mathbf{x}) := p_1(\mathbf{x})m(d\mathbf{x}), \dots, m_k(d\mathbf{x}) := p_k(\mathbf{x})m(d\mathbf{x})$.*

Example 1. *Suppose that we want to simulate a Poisson point process on the unit circle $S := \{(x, y) : x^2 + y^2 \leq 1\}$ with mean measure:*

$$m'(B) = \iint_{B\cap S} \sqrt{x^2 + y^2}dxdy.$$

A method to do this is based on the previous thinning method. Suppose that we generate a "homogeneous" Poisson point process M on the square $R := \{(x, y) : |x| \leq 1, |y| \leq 1\}$ with an intensity of $\lambda = 8$ points per unit area. That is, the mean measure of M is

$$m(B) = \frac{1}{4} \iint_B \lambda dxdy.$$

Let $\{(x_i, y_i)\}_i$ denote the atoms of the Poisson random measure M. Now, consider the following thinning process. We classify the point (x_i, y_i) of type 1 with probability $p(x_i, y_i) := \frac{1}{2}\sqrt{x_i^2 + y_i^2}$ and of type 2 with probability $1 - p(x_i, y_i)$. Suppose that $\{(x_i^1, y_i^1)\}_i$ are the point of type 1. Then, this process is a Poisson point process with mean measure m'.

3.1.2. Integration with respect to a Poisson random measure

Let M be a Poisson random measure as Definition 4. Since $M(\cdot; \omega)$ is an atomic random measure for each ω, say $M(\cdot; \omega) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i(\omega)}(\cdot)$, one can define the integral

$$M(f) := \int_S f(\mathbf{x})M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i),$$

for any measurable nonnegative deterministic function f. This is a $\bar{\mathbb{R}}_+ = \mathbb{R} \cup \{\infty\}$-valued r.v. such that

$$\mathbb{E}\left[e^{-\int f(\mathbf{x})M(d\mathbf{x})}\right] = \exp\left\{-\int \left(1 - e^{-f(\mathbf{x})}\right) m(d\mathbf{x})\right\}, \quad \mathbb{E}\left[\int f(\mathbf{x})M(d\mathbf{x})\right] = \int f(\mathbf{x})m(d\mathbf{x});$$

see [30, Lemma 10.2]. Also, if $B \in \mathcal{S}$ is such that $m(B) < \infty$, then

$$\int_B f(\mathbf{x})M(d\mathbf{x}) := \sum_{i:\mathbf{x}_i \in B} f(\mathbf{x}_i),$$

is a well-defined $\mathbb{R}^d-$valued r.v. for any measurable function $f : \mathcal{S} \to \mathbb{R}^d$. Its characteristic function is given by

$$\mathbb{E}\left[e^{\mathbf{i}\left\langle \int_B f(\mathbf{x})M(d\mathbf{x}), \mathbf{u}\right\rangle}\right] = \exp\left\{\int_B \left(e^{\mathbf{i}\langle f(\mathbf{x}),\mathbf{u}\rangle} - 1\right) m(d\mathbf{x})\right\}.$$

Furthermore, if $B_1, \dots, B_m$ are disjoints sets in $\mathcal{S}$ with finite measure, then

$$\int_{B_1} f(\mathbf{x}) M(d\mathbf{x}), \dots, \int_{B_m} f(\mathbf{x}) M(d\mathbf{x}).$$

are independent (see [46, Proposition 19.5]).

In the general case, determining conditions for the integral $\int_{\mathcal{S}} f(\mathbf{x}) M(d\mathbf{x})$ to be well-defined requires some care. Let us assume that m is a radon measure (that is, $m(K) < \infty$, for any compact $K \subset S$). Then, $\int_S f(\mathbf{x}) M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i)$ is well-defined for any bounded function $f : S \to \mathbb{R}$ of compact support. We say that the integral $\int_S f(\mathbf{x}) M(d\mathbf{x})$ exists if

$$\int_S f_n(\mathbf{x}) M(d\mathbf{x}) \xrightarrow{\mathbb{P}} X, \quad \text{as} \quad n \to \infty,$$

for a random variable X and any sequence f_n of bounded functions with compact support such that $|f_n| \leq |f|$ and $f_n \to f$. In that case, the so-called *Poisson integral* $\int_S f(\mathbf{x}) M(d\mathbf{x})$ is defined to be that common limit X. We define in a similar way the so-called *compensated Poisson integral of* f, denoted by $\int_S f(\mathbf{x})(M-m)(d\mathbf{x})$. The following theorem gives conditions for the existence of the Poisson integrals (see [30, Theorem 10.15]):

Proposition 4. *Let M be a Poisson random measure as in Definition 4. Then,*

(a) $M(f) = \int_S f(\mathbf{x}) M(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})| \wedge 1) m(d\mathbf{x}) < \infty$
(b) $(M - m)(f) := \int_S f(\mathbf{x})(M - m)(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})|^2 \wedge |f(\mathbf{x})|) m(d\mathbf{x}) < \infty$

3.2. The Lévy-Itô decomposition

The following result, called the Lévy-Itô decomposition, is fundamental for the theory of Lévy processes. It says that any Lévy process X is the superposition of a constant drift bt, a Brownian component ΣW_t, a compound Poisson process X_t^{cp}, and the limit of compensated Poisson processes. As stated below, it characterizes not only Lévy processes but also processes with independent increments (called *additive processes*).

Theorem 3. ***[13.4, Kallenberg]*** *Let $\{X_t\}_{t \geq 0}$ be a rcll process in $\mathbb{R}^d$ with $X(0) = 0$. Then, X has independent increments without fixed jumps times if and only if, there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$,*

$$X_t(\omega) = b_t(\omega) + G_t(\omega) + \int_0^t \int_{\{|x|>1\}} x \; M(\omega; ds, dx) + \int_0^t \int_{\{|x|\leq 1\}} x \; (M-m)(\omega; ds, dx), \quad (22)$$

for any $t \geq 0$, and for a continuous function b with $b_0 = 0$, a continuous centered Gaussian process G with independent increments and $G_0 = 0$, and an independent Poisson random *measure M on $[0,\infty)\times\mathbb{R}_0^d$ with mean measure m satisfying*

$$\int_0^t \int_{\mathbb{R}_0^d} (|x|^2 \wedge 1)\, m(ds,dx) < \infty, \quad \forall t > 0. \tag{23}$$

The representation is almost surely unique, and all functions b, processes G, and measures m with the stated properties may occur.

Note that the above theorem states that the *jump random measure* M_X of X, defined by

$$M_X((s,t]\times B) := \sum_{u\in(s,t]:\Delta X_u\neq 0} \mathbf{1}\{\Delta X_u \in B\},$$

is almost surely a Poisson process with mean measure $m(dt,dx)$. In the case of a Lévy process (that is, we also assume that X has stationary increments), the previous theorem implies that M_X is a Poisson random measure in $\mathbb{R}_+\times\mathbb{R}_0$ with mean measure $m(dt,dx)=\nu(dx)dt$, for a measure ν satisfying (10). In that case, the representation (22) takes the following form:

$$X_t = bt + \Sigma\, W_t + \int_0^t \int_{\{|x|>1\}} x\; M(ds,dx) + \int_0^t \int_{\{|x|\leq 1\}} x\; (M-m)(ds,dx), \tag{24}$$

where W is a d-dimensional Wiener process. The third term is a compound Poisson process with intensity of jumps $\nu(|x|>1)$ and jump distribution $\mathbf{1}_{\{|x|>1\}}\nu(dx)/\nu(|x|>1)$. Similarly, the last term can be understood as the limit of compensated Poisson processes as follows:

$$\int_0^t \int_{\{|x|\leq 1\}} x(M-m)(ds,dx) = \lim_{\varepsilon\downarrow 0} \int_0^t \int_{\{\varepsilon<x|\leq 1\}} x\; (M-m)(ds,dx). \tag{25}$$

Furthermore, the convergence in (25) is uniform on any bounded interval of t (c.f. [19.2, Sato]).

3.3. Some sample path properties

One application of the Lévy-Itô decomposition (24) is to determine conditions for certain path behavior of the process. The following are some cases of interest (see Section 19 in [46] for these and other path properties):

1. *Path-continuity*: The only continuous Lévy processes are of the form $bt+\sigma W_t$.
2. *Finite-variation*: A necessary and sufficient condition for X to have a.s. paths of bounded variation is that $\sigma=0$ and

$$\int_{\{|x|\leq 1\}} |x|\nu(dx) < \infty.$$

Note that in that case one can write

$$X_t = b_0 t + \int_0^t \int \; x \; M(ds, dx),$$

where $b_0 := b - \int_{|x|\leq 1} x\nu(dx)$, called the *drift* of the Lévy process, is such that

$$\mathbb{P}\left(\lim_{t\to 0} \frac{1}{t} X_t = b_0\right) = 1.$$

A process of finite-variation can be written as the difference of two non-decreasing processes. In the above representation, this processes will be $b_0 t + \int_0^t \int_{x>0} \; x \; M(ds, dx)$, and $\int_0^t \int_{x<0} \; x \; M(ds, dx)$ when $b_0 > 0$.

3. A non-decreasing Lévy process is called a *subordinator*. Necessary and sufficient conditions for X to be a subordinator are that $b_0 > 0$, $\sigma = 0$, and $\nu((-\infty, 0)) = 0$.

4. Simulation of Lévy processes

4.1. Approximation by skeletons and compound Poisson processes

Accurate path simulation of a pure jump Lévy processes $X = \{X(t)\}_{t\in[0,1]}$, regardless of the relatively simple statistical structure of their increments, present some challenges when dealing with *infinite jump activity* (namely, processes with infinite Lévy measure). One of the most popular simulation schemes is based on the generation of *discrete skeletons*. Namely, the discrete skeleton of X based on equally spaced observations is defined by

$$\widetilde{X}_t = \sum_{k=1}^{\infty} X_{\frac{k-1}{n}} \mathbf{1}_{\left[\frac{k-1}{n} \leq t < \frac{k}{n}\right]} = \sum_{k=1}^{\infty} \Delta_k \mathbf{1}_{\{t \geq \frac{k}{n}\}},$$

where $\Delta_k = X_{k/n} - X_{(k-1)/n}$ are i.i.d. with common distribution $\mathcal{L}(X_{1/n})$. Popular classes where this method is applicable are Gamma, variance Gamma, Stable, and Normal inverse Gaussian processes (see [18, Section 6.2]). Lamentably, the previous scheme has limited applications since in most cases a r.v. with distribution $\mathcal{L}(X_{1/n})$ is not easy to generate.

A second approach is to approximate the Lévy process by a finite-jump activity Lévy processes. That is, suppose that X is a pure-jump Lévy process, then, in light of the Lévy-Itô decomposition (24), the process

$$X_t^{0,\varepsilon} \equiv t\left(b - \int_{\{|x|\geq\varepsilon\}} x\,\nu(dx)\right) + \sum_{s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq\varepsilon\}} \tag{26}$$

converges uniformly on any bounded interval to X a.s. (as usual $\Delta X_t \equiv X_t - X_{t^-}$). The process $\sum_{s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq \varepsilon\}}$ can be simulated using a *compound Poisson process* of the form $\sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon$, where N_t^ε is a homogeneous Poisson process with intensity $\nu(|x|\geq\varepsilon)$ and $\{J_i^\varepsilon\}_{i=1}^\infty$ are i.i.d with common distribution $\nu_\varepsilon(dx) \equiv \mathbf{1}_{\{|x|\geq\varepsilon\}}\nu(dx)/\nu(|x|\geq\varepsilon)$. Clearly, such a scheme is unsatisfactory because all jumps smaller than ε are totally ignored. An alternative method of simulation approximates the small jumps with a Wiener motion.

4.2. Approximation of the small jumps of a Lévy processes

Consider a Lévy process with Lévy triple (σ^2, b, ν). Define the following processes:

$$X_t^{1,\varepsilon} := b_\varepsilon t + \sigma W_t + \int_0^t \int_{|x|\geq\varepsilon} xM(dx, ds),$$

where $b_\varepsilon = b - \int_{\varepsilon<|x|\leq 1} x\nu(dx)$ and M is the jump-measure of X (a posterior a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_0^d$ with mean measure $\nu(dx)dt$). Consider the following pure jump Lévy process

$$X_t^\varepsilon := X_t - X_t^{1,\varepsilon} = \int_0^t \int_{\{|x|<\varepsilon\}} x\left\{M(dx, ds) - \nu(dx)ds\right\}.$$

Also, consider the jump-diffusion model

$$X_t^{2,\varepsilon} := b_\varepsilon t + (\sigma^2 + \sigma^2(\varepsilon))^{1/2} W_t + \int_0^t \int_{|x|\geq\varepsilon} xN(dx, ds),$$

where $\sigma^2(\varepsilon) = \int_{|x|\leq\varepsilon} x^2\nu(dx)$. Rosinski and Asmussen [5] establish the following approximation method:

Theorem 4. *Suppose that ν has no atoms in a neighborhood of the origin. Then,*

(a) $\{\sigma^{-1}(\varepsilon)X_t^\varepsilon\}_{t\geq 0}$ *converges in distribution to a standard Brownian motion* $\{B(t)\}_{t\geq 0}$ *if and only if*

$$\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty. \tag{27}$$

(b) Under (27), it holds that

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}(X_t \leq x) - \mathbb{P}(X_t^{2,\varepsilon} \leq x)\right| \leq c\,\frac{\int_{|x|\leq\varepsilon} x^3\nu(dx)}{\sigma^3(\varepsilon)} \leq c\,\frac{\varepsilon}{\sigma(\varepsilon)}.$$

The first part of the above theorem provides a way to approximate the small-jumps component of X properly scaled by a Wiener process. Condition (27) can be interpreted as an assumption requiring that the size of the jumps of $\sigma^{-1}(\varepsilon)X_\varepsilon$ are asymptotically vanishing. Part (b) suggests that the distribution of certain Lévy processes (with infinite jump activity) can be approximated closely by the combination of a Wiener process with drift and a compound Poisson process.

4.3. Simulations based on series representations

Throughout, $X = \{X_t\}_{t\in[0,1]}$ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and without Brownian component (which can be simulated separately). Let $M := M_X$ be the jump measure of the process X, which we assumed admits the following representation:

Condition 1. *The following series representation holds:*

$$M(\cdot) = \sum_{i=1}^{\infty} \delta_{(U_i, H(\Gamma_i, V_i))}(\cdot)\,, \quad a.s. \tag{28}$$

for a homogeneous Poisson process $\{\Gamma_i\}_{i=1}^{\infty}$ on $\mathbb{R}_+$ with unit intensity rate, an independent random sample $\{U_i\}_{i=1}^{\infty}$ uniformly distributed on $(0,1)$, and an independent random sample $\{V_i\}_{i=1}^{\infty}$ with common distribution F on a measurable space S, and a measurable function $H : (0,\infty)\times S \to \mathbb{R}^d$.

Remark 3. *Representation (28) can be obtained (in law) if the Lévy measure has the decomposition*

$$\nu(B) = \int_0^{\infty} \sigma(u;B)du, \tag{29}$$

where $\sigma(u;B) = \mathbb{P}\left[H(u,\mathbf{V}) \in B\right]$. It is not always easy to obtain (29). The following are typical methods: the inverse Lévy measure method, Bondensson's method, and Thinning method (see Rosinski [43] for more details).

Define

$$A(s) = \int_0^s \int_S H(r,v) I\left(\|H(r,v)\| \le 1\right) F(dv)dr. \tag{30}$$

Condition 2.

$$A(\Gamma_n) - A(n) \to 0, \quad a.s. \tag{31}$$

Lemma 1. *The limit in (31) holds true if any of the following conditions is satisfied:*
i. $b \equiv \lim_{s\to\infty} A(s)$ *exists in* $\mathbb{R}^d$*;*
ii. *the mapping* $r \to \|H(r,v)\|$ *is nonincreasing for each* $v \in S$*.*

Proposition 1. *If the conditions 1 and 2 are satisfies then, a.s.*

$$X_t = bt + \sum_{i=1}^{\infty} \left(H(\Gamma_i, V_i) I\left(U_i \le t\right) - tc_i \right), \tag{32}$$

for all $t \in [0,1]$*, where* $c_i \equiv A(i) - A(i-1)$*.*

Remark 4. *The series (32) simplifies further when* $\int_{|x|\le 1} |x| \nu(d\mathbf{x}) < \infty$*, namely, when* X *has paths of bounded variation. Concretely, a.s.*

$$X_t = (b - a)\, t + \sum_{i=1}^{\infty} J_i \mathrm{I}\left(U_i \le t\right), \tag{33}$$

where $a = \int_{|x|\le 1} x\nu(dx)$*. The vector* $b_0 \equiv b - a$ *is the* drift *of the Lévy process.*

5. Density Transformation of Lévy processes

The following two results describe Girsanov-type theorems for Lévy processes. Concretely, the first result provides conditions for the existence of an equivalent probability measure under which X is still a Lévy process, while the second result provides the density process. These theorems have clear applications in mathematical finance as a device to define risk-neutral probability measures. The proofs can be found in Section 33 of Sato [46]. Girsanov-type theorems for more general processes can be found in Jacod and Shiryaev [29] (see also Applebaum [4] for a more accessible presentation).

Theorem 5. *Let* $\{X_t\}_{t\le T}$ *be a real Lévy process with Lévy triple* (σ^2, b, ν) *under some probability measure* $\mathbb{P}$*. Then the following two statements are equivalent:*

(a) There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ *such that* $\{X_t\}_{t\le T}$ *is a Lévy process with triplet* (σ'^2, b', ν') *under* $\mathbb{Q}$*.*

(b) All the following conditions hold.

  (i) $\nu'(dx) = k(x)\nu(dx)$*, for some function* $k : \mathbb{R} \to (0, \infty)$*.*

  (ii) $b' = b + \int x \mathbf{1}_{|x|<1}(k(x) - 1)\nu(dx) + \sigma\eta$*, for some* $\eta \in \mathbb{R}$*.*

  (iii) $\sigma' = \sigma$*.*

  (iv) $\int \left(1 - \sqrt{k(x)}\right)^2 \nu(dx) < \infty$*.*

Theorem 6. *Suppose that the equivalent conditions of the previous theorem are satisfied. Then, $\xi \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}$, is given by the formula*

$$\xi \equiv \exp\left(\eta\sigma W_T - \frac{1}{2}\eta^2\sigma^2 T \right. \\ \left. + \lim_{\varepsilon\downarrow 0}\left(\int_0^T \int_{|x|>\varepsilon} \log k(x) M(ds,dx) - T\int_{|x|>\varepsilon}(k(x)-1)\nu(dx)\right)\right),$$

with $\mathbb{E}_{\mathbb{P}}[\xi] \equiv 1$. The convergence on the right-hand side of the formula above is uniform in t on any bounded interval.

6. Exponential Lévy models

As it was explained in the introduction, the simplest extension of the GBM (1) is the Geometric or exponential Lévy model:

$$S_t = S_0 e^{X_t}, \tag{34}$$

where X is a general Lévy process with Lévy triplet (σ^2, b, ν) defined on a probability space $(\Omega, \mathbb{P})$. In this part, we will review the financial properties of this model. As in the Black-Scholes model for option pricing, we shall also assume the existence of a risk-free asset B with constant interest rate r. Concretely, B is given by any of the following two equivalent definitions:

$$\begin{array}{c} dB_t = rB_t dt \\ B_0 = 1 \end{array}, \qquad \text{or} \qquad B_t = e^{rt}. \tag{35}$$

The following are relevant questions: (1) Is the market arbitrage-free?; (2) Is the market complete?; (3) Can the arbitrage-free prices of European simple claim $\mathcal{X} = \Phi(S_T)$ be computed in terms of a Black-Scholes PDE?.

6.1. Stochastic integration and self-financing trading strategies

As in the classical Black-Scholes model, the key concept to define arbitrage opportunities is that of a self-financing trading strategy. Formally, this concept requires the development of a theory of stochastic integration with respect to Lévy processes and related processes such as (34). In other words, given a suitable trading strategy $\{\beta_t\}_{0\le t\le T}$, so that β_t represents the number of shares of the stock held at time t, we want to define the integral

$$G_t := \int_0^t \beta_u dS_u, \tag{36}$$

which shall represent the net gain/loss in the stock at time t. Two different treatments of the general theory of stochastic integration with respect to semimartingales can be found in Jacod and Shiryaev [29], and Protter [41]. More accessible presentations of the topic are given in, e.g., Applebaum [4], and Cont and Tankov [18]. Our goal in this part is only to recall the general ideas behind (36) and the concept of self-financibility.

We first note that the process β should not only be adapted to the information process $\{\mathcal{F}_t\}_{0\leq t\leq T}$ generated by the stock price (i.e. β_t should depend only on the stock prices up to time t), but also should be *predictable*, which roughly speaking means that its value at any time t can be determined from the information available right before t. As usual, (36) can be defined for simple trading strategies in a natural manner and then, this definition can be extended to a certain class of processes β as the limits of stochastic integrals for simple trading strategies. Concretely, consider a "buy-and-hold" trading strategy of the form $\beta_t := \mathbf{1}_{\{\tau_1<t\leq\tau_2\}}$, for deterministic times $0\leq\tau_1<\tau_2\leq T$. That is, β_t represents a strategy that buys one share of the stock "right-after" time τ_1 and holds it until time τ_2. Then, the net gain/loss process is $G_t=\int_0^t\beta_u dS_u = S_{\tau_2\wedge t}-S_{\tau_1\wedge t}$. Combinations of buy and hold strategies can be defined similarly as

$$\beta_t := \xi_0\mathbf{1}_{\{t=0\}}+\sum_{i=1}^n \xi_i\mathbf{1}_{\{\tau_{i-1}<t\leq\tau_i\}}, \tag{37}$$

where $0=\tau_0<\tau_1<\cdots<\tau_n\leq T$ are deterministic trading times and the value of ξ_i is revealed at time τ_{i-1}, for $i=1,\dots,n$, while ξ_0 is deterministic. The net gain/loss of the strategy (37) at time t is then given by

$$G_t=\int_0^t \beta_u dS_u = \xi_0 S_0+\sum_{i=1}^n \xi_i\left(S_{\tau_i\wedge t}-S_{\tau_{i-1}\wedge t}\right).$$

The integral (36) can subsequently be defined for more general processes β that can be approximated by simple processes of the form (37). For instance, if β is an adapted process (thus, for any $t\geq 0$, the value of β_t is revealed at time t) having paths that are left-continuous with right limits (lcrl), then for any sequence $0=\tau_0^n<\tau_1<\cdots<\tau_n^n=T$ such that $\max_k\left(\tau_k^n-\tau_{k-1}^n\right)\to 0$, it holds that

$$\beta_0 S_0+\sum_{i=1}^n \beta_{\tau_{i-1}}\left(S_{\tau_i\wedge t}-S_{\tau_{i-1}\wedge t}\right)\xrightarrow{\mathbb{P}}\int_0^t\beta_u dS_u,$$

as $n\to\infty$, where the convergence is uniform in $[0,T]$. The times τ_i^n can be taken to be *stopping times*, which means that at any time t, one can decide whether the event $\tau_i^n\leq t$ occurs or not.

Once a trading strategy has been defined, one can easily define a *self-financing strategy* on the market (34-35), as a pair (α, β) of adapted processes such that the so-called value process $V_t := \alpha_t B_t + \beta_t S_t$, satisfies that

$$V_t = V_0 + \int_0^t \alpha_u B_u r du + \int_0^t \beta_u dS_u,$$

or equivalently expressed in "differential form",

$$dV_t = \alpha_t B_t r dt + \beta_t dS_t.$$

Intuitively, the change of the portfolio value dV_t during a small time interval $[t, t+dt]$ is due only to the changes in the value of the primary assets in the portfolio and not due to the infusion or withdrawal of money into the portfolio.

6.2. Conditions for the absence of arbitrage

Let us recall that an arbitrage opportunity during a given time horizon $[0, T]$ is just a self-financing trading strategy (α, β) such that its value process $\{V_t\}_{0\leq t\leq T}$ satisfies the following three conditions:

$$(i) \quad V_0 = 0, \quad (ii) \quad V_T \geq 0, \quad a.s. \quad (iii) \quad \mathbb{P}(V_T > 0) > 0.$$

According to the first fundamental theorem of finance, the market (34-35) is arbitrage-free if there exists an *equivalent martingale measure* (EMM) $\mathbb{Q}$; that is, if there exists a probability measure $\mathbb{Q}$ such that the following two conditions hold:

(a) $\mathbb{Q}(B) = 0$ if and only if $\mathbb{P}(B) = 0$;
(b) The discounted price process $S_t^* := B_t^{-1} S_t$, for $0 \leq t \leq T$, is a martingale under $\mathbb{Q}$.

In order to find conditions for the absence of arbitrage, let us recall that for any function k satisfying (iv) in Theorem 5, and any real $\eta \in \mathbb{R}$, it is possible to find a probability measure $\mathbb{Q}^{(\eta,k)}$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}^{(\eta,k)}$, X is a Lévy process with Lévy triplet (σ^2, b', ν') given as in Theorem 5. Thus, under $\mathbb{Q}^{(\eta,k)}$, the discounted stock price S^* is also an exponential Lévy model

$$S_t^* = S_0 e^{X_t^*},$$

with X^* being a Lévy process with Lévy triplet $(\sigma^2, b' - r, \nu')$. It is not hard to find conditions for an exponential Lévy model to be a martingale (see, e.g., Theorem 8.20 in [18]). Concretely, S^* is a martingale under $\mathbb{Q}^{(\eta,k)}$ if and only if

$$b + \int x\mathbf{1}_{\{|x|\leq 1\}}(k(x) - 1)\nu(dx) + \sigma\eta - r + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^x - 1 - x\mathbf{1}_{\{|x|\leq 1\}})k(x)\nu(dx) = 0. \quad (38)$$

It is now clear that if $\nu \equiv 0$, there will exist a unique EMM of the form $\mathbb{Q}^{(\eta,k)}$, but if $\nu \neq 0$, there will exist in general infinitely-many of such EMM. In particular, we conclude that the exponential Lévy market (34-35) is incomplete. One popular EMM for exponential Lévy models is the so-called Esscher transform, where $k(x) = e^{\theta x}$, and η is chosen to satisfy (38).

6.3. Option pricing and integro-partial differential equations

As seen in the previous part, the exponential Lévy market is in general incomplete, and hence, options are not superfluous assets whose payoff can be perfectly replicated in an ideal frictionless market. The option prices are themselves subject to modeling. It is natural to adopt an EMM that preserve the Lévy structure of the log return process $X_t = \log(S_t/S_0)$ as in the previous section. From now on, we adopt exactly this option pricing model and assume that the time-t price of a European claim with maturity T and payoff $\mathcal{X}$ is given by

$$\Pi_t = \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(T-t)}\mathcal{X} \middle| S_u, u \leq t \right\},$$

where $\mathbb{Q}$ is an EMM such that

$$S_t = S_0 e^{rt + X_t^*},$$

with X^* being a Lévy process under $\mathbb{Q}$. Throughout, (σ^2, b^*, ν^*) denotes the Lévy triplet of X^* under $\mathbb{Q}$.

Note that in the case of a simple claim $\mathcal{X} = \Phi(S_T)$, there exists a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Pi_t := C(t, S_t(\omega))$. Indeed, by the Markov property, one can easily see that

$$C(t,x) = e^{-r(T-t)}\mathbb{E}_Q \left[\Phi \left(x e^{r(T-t) + X^*_{T-t}} \right) \right]. \tag{39}$$

The following theorem shows that C satisfies an integro-partial differential equation (IPDE). The IPDE equation below is well-known in the literature (see e.g. [15] and [42]) and its proof can be found in, e.g., [18, Proposition 12.1].

Proposition 5. *Suppose the following conditions:*

1. $\int_{|x|\geq 1} e^{2x}\nu^*(dx) < \infty$;
2. *Either* $\sigma > 0$ *or* $\liminf_{\varepsilon \searrow 0} \varepsilon^{-\beta} \int_{|x|\leq\varepsilon} |x|^2 \nu^*(dx) < \infty$.
3. $|\Phi(x) - \Phi(y)| \leq c|x-y|$, *for all* x, y *and some* $c > 0$.

Then, the function $C(t,x)$ *in (39) is continuous on* $[0,T] \times [0,\infty)$, $C^{1,2}$ *on* $(0,T) \times (0,\infty)$ *and verifies the integro-partial differential equation:*

$$\begin{aligned}
&\frac{\partial C(t,x)}{\partial t} + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(t,x) - rC(t,x) \\
&+ \int_{\mathbb{R}_0} \left(C(t, xe^y) - C(t,x) - x(e^y - 1)\frac{\partial C}{\partial x}(t,x) \right) \nu^*(dy) = 0,
\end{aligned}$$

on $[0, T) \times (0, \infty)$ *with terminal condition* $C(T, x) = \Phi(x)$*, for all* x*.*

References

[1] Y. Ait-Sahalia and J. Jacod. Testing for jumps in a discretely observed process. Technical report, Princeton University and Université de Paris VI, September 2006.

[2] Y. Aït-Sahalia, P.A. Mykland, and L. Zhang. How often to sample a continuous-time process in the presence of market microstructure noise. *The review of financial studies*, 18(2), February 2005.

[3] T. Andersen and L. Benzoni. Stochastic Volatility. Technical report, Working paper, Northwestern University, 2007.

[4] D. Applebaum. *Lévy processes and stochastic calculus.* Cambridge Univ. Press, 2004.

[5] S. Asmussen and J. Rosiński. Approximations of small jumps of Lévy processes with a view towards simulation. *Journal of Applied Probability*, 38:482–493, 2001.

[6] O.E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2:41–68, 1998.

[7] O.E. Barndorff-Nielsen and N. Shephard. Modelling by Lévy processess for financial economics. *Lévy Processes. Theory and Applications, by O.E. Barndorff-Nielsen, T. Mikosch, and S.I. Resnick*, pages 283–318, 2001.

[8] O.E. Barndorff-Nielsen and N. Shephard. Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics*, 4(1):1–30, 2006.

[9] O.E. Barndorff-Nielsen and N. Shephard. Variation, jumps, Market Frictions and High Frequency Data in Fnancial Econometric. *In Advances in Economics and Econometrics. Theory and Applications, by R. Blundell, T. Persson, and W. Newey (Eds.)*, 2007.

[10] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–659, 1973.

[11] P. Carr, H. Geman, D. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *Journal of Business*, pages 305–332, April 2002.

[12] P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.

[13] P. Carr, D. Madan, and E. Chang. The variance Gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.

[14] P. Carr and L. Wu. Time-Changed Levy Processes and Option Pricing. *Journal of Financial Economics*, 71(1):113–141, 2004.

[15] T. Chan. Pricing contingent claims on stocks driven by Lévy processes. *The Annals of Applied Probability*, 9(2):504–528, 1999.

[16] F. Comte and V. Genon-Catalot. Nonparametric adaptive estimation for pure jump Lé vy processes. *Working paper, arXiv:0806.3371.2008*, 2008.

[17] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1:223–236, 2001.

[18] R. Cont and P. Tankov. *Financial modelling with Jump Processes.* Chapman & Hall, 2004.

[19] E. Eberlein. Application of Generalized Hyperbolic Lévy Motions to Finance. *Lévy Processes. Theory and Applications, by O.E. Barndorff-Nielsen, T. Mikosch, and S.I. Resnick*, pages 319–336, 2001.

[20] E. Eberlein and U. Keller. Hyperbolic Distribution in Finance. *Bernoulli*, 1:281–299, 1995.

[21] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. New York, Wiley, 1968.

[22] J.E. Figueroa-López. *Nonparametric estimation of Lévy processes with a view towards mathematical finance.* PhD thesis, Georgia Institute of Technology, April 2004. http://etd.gatech.edu. No. etd-04072004-122020.

[23] J.E. Figueroa-López. Small-time moment asymptotics for Lévy processes. *Statistics and Probability Letters*, 78:3355–3365, 2008.

[24] J.E. Figueroa-López. Nonparametric estimation for Lévy models based on discrete-sampling. *IMS Lecture Notes - Monograph Series. Optimality: The Third Erich L. Lehmann Symposium*, 57:117–146, 2009.

[25] J.E. Figueroa-López and C. Houdré. Risk bounds for the non-parametric estimation of Lévy processes. *IMS Lecture Notes - Monograph Series. High Dimensional Probability*, 51:96–116, 2006.

[26] S. Gugushvili. Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process. *Working paper, arXiv:0807.3469v1.2008*, 2008.

[27] J. Jacod. Asymptotic property of realized power variations and related power variations and related functionals of semimartingales. *Preprint*, 2006.

[28] J. Jacod. Asymptotic properties of power variations of Lévy processes. *ESAIM:P&S*, 11:173–196, 2007.

[29] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 2003.

[30] O. Kallenberg. *Foundations of Modern Probability.* Springer-Verlag, Berlin, New York, Heidelberg, 1997.

[31] V.M. Kruglov. A note on infinitely divisible distributions. *Theory of Probability and Applications*, 15:319–324, 1970.

[32] A. Kyprianou, W. Schoutens, and P. Wilmott. *Exotic option pricing and advanced Lévy models.* Wiley, 2005.

[33] C. Mancini. Non parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics*, 36:270–296, 2009. Available at [ArXiv math.ST/0607378].

[34] B. Mandelbrot. The variation of certain speculative prices. *The Journal of Business*, 36:394–419, 1963.

[35] R.C. Merton. Option pricing when underlying stock returns are discontinuous. *J. Financial Economics*, 3:125–144, 1976.

[36] M. Neumann and M. Reiss. Nonparametric estimation for Lévy processes from low-frequency observations. *Working paper, ArXiv:0709.2007v1*, 2007.

[37] M. Podolskij. *New Theory on estimation of integrated volatility with applications*. PhD thesis, Ruhr-Universität Bochum, April 2006. Available on the web.

[38] M. Podolskij and M. Vetter. Estimation of volatility functionals in the simultaneous presence of microstructure noise. Technical report, 2007. Bernoulli (Forthcoming).

[39] M. Podolskij and M. Vetter. Bipower-type estimation in a noisy diffusion setting. Technical report, 2009. Available online.

[40] S.J. Press. A compound event model for security prices. *The Journal of Business*, 40:317–335, 1967.

[41] P. Protter. *Stochastic Integration and Differentil Equations*. Springer, 2004. 2nd Edition.

[42] S. Raible. *Lévy processes in Finance: Theory, Numerics, and Empirical Facts*. PhD thesis, Albert-Ludwigs-Universitat Freiburg, January 2000.

[43] J. Rosiński. Series representations of Lévy processes from the perspective of point processes. *In Lévy processes-Theory and Applications*, pages 401–415, 2001.

[44] J. Rosiński. Tempering stable processes. *Stochastic processes and their applications*, 117:677–707, 2007.

[45] P. Samuelson. Rational theory of warrant pricing. *Industrial management review*, (6):13–32, 1965.

[46] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

[47] N. Shephard. General Introduction. *In N. Shephard (Ed.), Stochastic Volatility*, 2005.

[48] V. Todorov. Econometric analysis of jump-driven stochastic volatility models. *Forthcoming in Journal of Econometrics*, 2008.

[49] R.N. Watteel and R.J. Kulperger. Nonparametric estimation of the canonical measure for innitely divisible distributions. *Journal of Statistical Computation and Simulation*, 73(7):525–542, 2003.

[50] J. Woerner. Variational sums and power variation: a unifying approach to model selection and estimation in semimartingale models . *Statistics and Decisions*, 21:47–68, 2003.

[51] J. Woerner. Power and multipower variations: inference for high frequency data. *In Stochastic Finance, A.N. Shiryaev, M. do Rosário Grosshino, P. Oliviera, M. Esquivel, eds.*, pages 343–354, 2006.